GEOMETRY II: HOMEWORK 3

DUE MARCH 13

A local trivialization, $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^k$, is equivalent to local trivializing sections, $\{\mathfrak{s}_\mu\}_{\mu=1}^k$. Each section $\mathfrak{s}_\mu$ corresponds to the standard basis $\mathfrak{e}_\mu$ for $\mathbb{R}^k$. Given a connection ∇, $\nabla\mathfrak{s}_\nu$ can be expressed as a linear combination of $\{\mathfrak{s}_\mu\}_{\mu=1}^k$, with the coefficients being 1-forms on $\mathcal{U}$. Namely,

$$\nabla\mathfrak{s}_\nu = \sum_{\mu=1}^{k} \omega_\nu^\mu \otimes \mathfrak{s}_\mu \quad \text{where } \omega_\nu^\mu \in \Omega^1(\mathcal{U}) \ .$$

The expression is a section of $T^*M \otimes E$ over $\mathcal{U}$. Sometimes $\otimes$ is omitted.

Any local section can be expressed as $\sum_{\mu=1}^k \alpha^\mu\, \mathfrak{s}_\mu$ where $\alpha^\mu \in \mathcal{C}^\infty(\mathcal{U})$. Due to the properties of a connection,

$$\begin{aligned}\nabla(\sum_{\mu=1}^{k} \alpha^\mu\, \mathfrak{s}_\mu) &= \sum_{\mu=1}^{k}(\mathrm{d}\alpha^\mu)\mathfrak{s}_\mu + \sum_{\mu=1}^{k} \alpha^\mu \nabla\mathfrak{s}_\mu \\ &= \sum_{\mu=1}^{k}(\mathrm{d}\alpha^\mu)\mathfrak{s}_\mu + \sum_{\mu,\nu}^{k} \alpha^\nu \omega_\nu^\mu \mathfrak{s}_\mu = \sum_{\mu=1}^{k}(\sum_{\nu=1}^{k} \mathrm{d}\alpha^\mu + \omega_\nu^\mu\, \alpha^\nu)\mathfrak{s}_\mu \ .\end{aligned}$$

That is to say, ∇ in terms of the trivialization is $\mathrm{d} + [\omega_\nu^\mu]$ acting on $\mathbb{R}^k$-valued functions.

(1) Endow E a bundle metric. A connection ∇ is called a *metric connection* if

$$\mathrm{d}\langle\mathfrak{s}, \tilde{\mathfrak{s}}\rangle = \langle\nabla\mathfrak{s}, \tilde{\mathfrak{s}}\rangle + \langle\mathfrak{s}, \nabla\tilde{\mathfrak{s}}\rangle$$

for any two $\mathfrak{s}, \tilde{\mathfrak{s}} \in \Gamma(E)$. Prove that a metric connection always exists. (Just do the real vector bundle case.)

(2) Suppose that E is a real vector bundle with a bundle metric and a metric connection. In terms of an *orthonormal*, local trivializing sections, what can you say about ω_ν^μ?

What about the complex case with the Hermitian metric?

(3) For those two examples in the lecture (regular surface case, and tautological bundle over $\mathbb{CP}^1$), is the connection a metric connection?

Basically, ω_ν^μ is given on some chart during the lecture. Do they obey the condition you found in part (2)?