# GEOMETRY II: HOMEWORK 2

DUE MARCH 6

(1) Polar Decomposition for $\mathrm{GL}(n;\mathbb{R})$.

(a) Show that any $A \in \mathrm{GL}(n;\mathbb{R})$ has a unique decomposition as $A = OP$ where $O$ is orthogonal, and $P$ is symmetric, positive-definite.

(b) The space of all $n \times n$ symmetric matrices can be identified with $\mathbb{R}^{\frac{n(n+1)}{2}}$. Positive-definite ones form an open subset in it. Prove that the space of all $n \times n$ symmetric, positive-definite matrices is contractible.

(Check the definition of "contractible" from wikipedia or any topology textbook. A "contractible" space is usually regarded as having "trivial topology".)

Think about the case of $\mathrm{GL}(n;\mathbb{C})$, and you do not need to submit this part.

(2) Suppose that $\mathbb{R}^k \to E \xrightarrow{\pi} M$ is a vector bundle. Let

$$E \times_M E = \{(e_1, e_2) \in E \times E \mid \pi(e_1) = \pi(e_2)\} \ .$$

Namely, it associates $E_p \times E_p$ for every $p \in M$. Locally, $E|_{\mathcal{U}} = \mathcal{U} \times \mathbb{R}^k$, $E|_{\mathcal{U}} \times E|_{\mathcal{U}} = (\mathcal{U} \times \mathbb{R}^k) \times (\mathcal{U} \times \mathbb{R}^k)$, and $(E \times E)|_{\mathcal{U}} = \{((x,u),(x,v))\}$.

A bundle metric is a smooth map $\mathfrak{g} : E \times_M E \to \mathbb{R}$ which defines a inner product on $E_p$ for every $p$.

(a) Show that any (real) vector bundle always admits a bundle metric. (But it is never unique.) You may need the partition of unity of $M$.

(b) Prove that for any (real) vector bundle, the transition functions can be required to be orthogonal matrices, i.e.

$$g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{O}(k) \subset \mathrm{GL}(k;\mathbb{R}) \ .$$

Hints: Suppose that $\{s_j\}_{j=1}^k$ form local trivializing sections over some open set. Performing Gram-Schmidt process gives pointwise orthonormal sections over the same open set.

(c) Show that any real vector bundle is isomorphic (abstractly) to its dual bundle.

(3) Do (2.a) and (2.b) for complex vector bundles and Hermitian inner product.

Think about what happens for part (c): everything is over $\mathbb{C}$. Will $E$ isomorphic to $E^*$ over $\mathbb{C}$? You do not have to submit this part.