GEOMETRY II: HOMEWORK 1

DUE FEBRUARY 27

(1) Prove that the tautological line bundle over $\mathbb{RP}^n$ is not a trivial bundle. Recall that $\mathbb{RP}^n = \mathbf{S}^n / \pm 1$.

(2) The Grassmannian $\mathrm{Gr}(k, n)$, the space of all k-planes in $\mathbb{R}^n$, is a smooth manifold of dimension $k(n-k)$.

   (a) Define analogously the tautological vector bundle E over the Grassmannian $\mathrm{Gr}(k, n)$.

   (b) Prove that E is a smooth manifold of dimension $k(n-k+1)$.

   (c) For $\mathrm{Gr}(2, 4)$, work out its local trivializations over two coordinate charts, and find the transition functions $g_{\mathcal{UV}}$.

(3) Consider $\mathbb{CP}^1 = \{\text{complex lines in } \mathbb{C}^2\}$. It is the one-point compactification of $\mathbb{C}$, and is diffeomorphic to $\mathbb{S}^2$.

   (a) Define analogously the tautological (complex) line bundle E over $\mathbb{CP}^1$.

   (b) Recall that $\mathbb{CP}^1 = \dfrac{\mathbb{C} \cup \mathbb{C}}{z \sim w = z^{-1}}$. In terms of this coordinate cover, work out the transition function of the tautological bundle.