recall $d : \Omega^j(M) \to \Omega^{j+1}(M)$, $d^2 = 0$

de Rham cohomology $H^j_{dR}(M) = \ker(d : \Omega^j \to \Omega^{j+1}) / \operatorname{im}(d : \Omega^{j-1} \to \Omega^j)$

With a metric on M, good representative of the cohomology class?

§I. linear algebra

In this section, all the spaces are finite dimensional vector spaces over $\mathbb{R}$ with an inner product

1° $V \xrightarrow{A} W$, A: linear transform

$V = \ker A \oplus (\ker A)^\perp$, $W = \operatorname{im} A \oplus (\operatorname{im} A)^\perp$

- $\langle Av, w \rangle = \langle v, A^T w \rangle \Rightarrow (\ker A)^\perp = \operatorname{im} A^T$ $\qquad V \xleftarrow{A^T} W$

$\Rightarrow V = \ker A \oplus \operatorname{im} A^T$, $W = \operatorname{im} A \oplus \ker A^T$

- $A^T A : V \circlearrowleft$ $\quad \langle A^T A v, v \rangle = \langle Av, Av \rangle$

$\Rightarrow \ker A^T A = \ker A$

$\operatorname{im} A^T$ can be decompose into eigenspaces of $A^T A$

- $(\ker A)^\perp = \operatorname{im} A^T \cong \operatorname{im} A$: direct sum of eigenspaces of $A^T A$ or $A A^T$ with nonzero eigenvalue

2° $0 \to V^0 \xrightarrow{A_0} \cdots \to V^j \xrightarrow{A_j} V^{j+1} \xrightarrow{A_{j+1}} \cdots \to V^n \to 0$

Suppose that $A_{j+1} A_j = 0 \quad \forall j$

$\Rightarrow$ form cohomology : $H^j(V, A) = \ker A_j / \operatorname{im} A_{j-1}$

question in this toy model : good representative of the cohomology class?

(i) $V^{j-1} \underset{A_{j-1}^*}{\overset{A_{j-1}}{\rightleftarrows}} V^j \underset{A_j^*}{\overset{A_j}{\rightleftarrows}} V^{j+1}$

(use the $*$ notation here: $\langle A_j v_j, \tilde{v}_{j+1} \rangle = \langle v_j, A_j^* \tilde{v}_{j+1} \rangle$ in matrix: transpose)

From 1°. $V^j = \ker A_j \oplus \operatorname{im} A_j^*$

$= \operatorname{im} A_{j-1} \oplus \ker A_{j-1}^*$ (with $\operatorname{im} A_{j-1} \subset \ker A_j$, $\ker A_{j-1}^* \supset \operatorname{im} A_j^*$)

Compare this two decomposition:

$A_j A_{j-1} = 0 \Rightarrow \operatorname{im} A_{j-1} \subset \ker A_j$

(Also, $A_{j-1}^* A_j^* = 0 \Rightarrow \operatorname{im} A_j^* \subset \ker A_{j-1}^*$)

$\Rightarrow \ker A_j = (\ker A_j \cap \ker A_{j-1}^*) \oplus \operatorname{im} A_{j-1}$

Hence, $V^j = \underbrace{(\ker A_j \cap \ker A_{j-1}^*) \oplus \operatorname{im} A_{j-1}}_{\ker A_j} \oplus \operatorname{im} A_j^*$

$\Rightarrow H^j = \ker A_j / \operatorname{im} A_{j-1} \cong \ker A_j \cap \ker A_{j-1}^* =: \mathcal{H}^j \subset V^j$

(ii) $\mathcal{H}^j = \ker A_j \cap \ker A_{j-1}^*$

recall that $\ker A_j = \ker A_j^* A_j$, $\ker A_{j-1}^* = \ker A_{j-1} A_{j-1}^*$

$A_{j-1}A_{j-1}^* \circlearrowright V^j \circlearrowleft A_j^* A_j$ $\quad \langle (A_j^* A_j + A_{j-1} A_{j-1}^*) v, v \rangle_{V^j} = 0$

$= |A_j v|^2_{V^{j+1}} + |A_{j-1}^* v|^2_{V^{j-1}}$

$\Rightarrow \mathcal{H}^j = \ker(A_j^* A_j + A_{j-1} A_{j-1}^*)$

(iii) $V^j \xleftarrow{A_j^*} V^{j+1}$

$V^j = \operatorname{im} A_{j-1} \oplus \mathcal{H}^j \oplus \operatorname{im} A_j^*$, with $\operatorname{im} A_{j-1}^* \overset{A_{j-1}}{\xrightarrow{\sim}} \operatorname{im} A_{j-1}$

$V^{j+1} = \operatorname{im} A_j \oplus \mathcal{H}^{j+1} \oplus \operatorname{im} A_{j+1}^*$, with $\operatorname{im} A_j^* \overset{A_j^*}{\xleftarrow{\sim}} \operatorname{im} A_j$ ($\mathcal{H}^{j+1} \oplus \operatorname{im} A_{j+1}^*$ annihilated by A_j^*)

$\operatorname{im} A_{j-1}^* \cong \operatorname{im} A_{j-1} \cong \bigoplus_{\lambda \neq 0}$ eigenspaces of $A_{j-1} A_{j-1}^*$

$\operatorname{im} A_j^* \cong \operatorname{im} A_j \cong \bigoplus_{\lambda \neq 0}$ eigenspaces of $A_j^* A_j$

$A_j^* A_j v = \lambda v \quad \lambda \neq 0 \quad \Rightarrow v = A_j^* \left(\frac{1}{\lambda} A_j v \right) \in \operatorname{im} A_j^*$

$\Rightarrow 0 = \underbrace{A_{j-1}^* A_j^*}_{= 0} A_j v = \lambda A_{j-1}^* v \Rightarrow v \in \ker A_{j-1}^*$

Thus, $(A_j^* A_j + A_{j-1} A_{j-1}^*) v = \lambda v$

<u>Upshot</u> $\operatorname{im} A_{j-1} \oplus \operatorname{im} A_j^* = \bigoplus_{\lambda \neq 0}$ eigenspaces of $A_{j-1}A_{j-1}^* + A_j^* A_j$

§II. Hodge theory : adjoint of d

(M, g) M^n: compact without boundary, oriented

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \longrightarrow 0$$

<u>thm</u> (Hodge) The finite dimensional story holds in this case as well.

<u>setting</u> What is the inner product?
What is the adjoint operator of d, d^*?

$0°$ $\iota(V_\alpha)\,d\mathrm{vol}$ $\quad \Omega^{n-1}(M) \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^n(M)$ $\quad f\, d\mathrm{vol} = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n$

$\mathfrak{X}(M) \ni V_\alpha \longleftrightarrow \alpha \in \Omega^1(M) \underset{\tilde{d}^*}{\overset{\tilde{d}}{\rightleftarrows}} \Omega^0(M) \ni f$

$\langle V_\alpha, W \rangle = \alpha(W)$

$$\alpha = \alpha_j\, dx^j \Rightarrow V_\alpha = g^{ij}\alpha_j \frac{\partial}{\partial x^i} = \alpha^i \frac{\partial}{\partial x^i}$$

$$\Rightarrow \iota(V_\alpha)\, d\mathrm{vol} = (-1)^{j-1}\alpha^j \sqrt{\det g}\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$$

What is $\tilde{d}$? $\quad d(\iota(V_\alpha)\, d\mathrm{vol}) = \frac{\partial}{\partial x^j}(\alpha^j \sqrt{\det g})\, dx^1 \wedge \cdots \wedge dx^n$

$$= \cdots = \left(\frac{\partial \alpha^j}{\partial x^j} + \Gamma^j_{jk}\alpha^k\right)\sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n$$

$$= \alpha^j{}_{;j}\, d\mathrm{vol} \quad \Rightarrow \tilde{d}\alpha = \alpha^j{}_{;j}$$

$$\begin{cases} \langle \alpha, \beta \rangle_{L^2} = \int g^{ij}\alpha_i \beta_j \, d\mathrm{vol} \\ \langle f, h \rangle_{L^2} = \int f h \, d\mathrm{vol} \end{cases}$$

$$\langle \tilde{d}\alpha, f \rangle_{L^2} = \int \alpha^j{}_{;j} f \, d\mathrm{vol}$$

$$\operatorname{div}\left(\alpha^j f \frac{\partial}{\partial x^j}\right) = \alpha^j{}_{;j} f + \alpha^j f_{;j} \qquad \left(= g^{ij}\alpha_i f_{;j}\right)$$

$$= -\int \langle \alpha, df \rangle \, d\mathrm{vol} = \langle \alpha, -df \rangle_{L^2}$$

<u>Upshot</u> Under this isomorphism, d^* on $\Omega^n(M)$ is equivalent to $-d$ acting on $\Omega^0(M) = \{\text{functions}\}$.

This phenomenon turns out to be true for $\Omega^k(M)$.

1° inner product on $\Omega^k(M)$, isomorphism between $\Omega^k(M)$ & $\Omega^{n-k}(M)$

↖ new ingredient

V : n-dimensional vector space, with inner product and orientation

(i) $\Lambda^k V$ inherits an inner product :

if $\{e_1, \cdots, e_n\}$: orthonormal basis for V

then let $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{i_1 < \cdots < i_k}$: orthonormal basis for $\Lambda^k V$

(ii) $\dim \Lambda^k V = \binom{n}{k} = \dim \Lambda^{n-k} V$ further assume $e_1 \wedge \cdots \wedge e_n$ is the orientation

$e_{i_1} \wedge \cdots \wedge e_{i_k} \longmapsto \pm e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$

where $\{i_1, \cdots, i_k, j_1 \cdots j_{n-k}\} = \{1, \cdots, n\}$
even permutation : $+1$ (odd : -1)

(iii) denote the isomorphism by $* : \Lambda^k V \longrightarrow \Lambda^{n-k} V$ (Hodge star operator)

<u>lemma</u>

- $** = (-1)^{k(n-k)}$
- $\langle \alpha, \beta \rangle = *(\alpha \wedge * \beta) \in \Lambda^0 V = \mathbb{R}$ (where α is a k-form and $*\beta$ an $(n-k)$-form, total degree n)

proof : DIY In fact, these properties characterise $*$

(iv) Apply to $V = T_p^* M \rightsquigarrow * : \Lambda^k T_p^* M \longrightarrow \Lambda^{n-k} T_p^* M$ for every $p \in M$

2° Note that for $f \in \Omega^0(M)$, $*f = f \, d\text{vol} \in \Omega^n(M)$

$**f = f$

(i) $\alpha, \beta \in \Omega^k(M) = \Gamma(M; \Lambda^k T^* M)$

At any $p \in M$, $\langle \alpha_p, \beta_p \rangle = *(\alpha_p \wedge * \beta_p) \in \mathbb{R}$

Define $\langle \alpha, \beta \rangle_{L^2} = \int *(\alpha \wedge * \beta) \, dvol$

$$= \int_M \alpha \wedge * \beta$$

(ii) $\Omega^{k-1}(M) \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^k(M)$

$\beta \qquad \alpha$

$$\langle d\beta, \alpha \rangle_{L^2} = \int_M (d\beta) \wedge * \alpha$$

$d(\beta \wedge * \alpha)$
$= d\beta \wedge * \alpha + (-1)^{k-1} d * \alpha$

$$= \int_M \beta \wedge ((-1)^k d * \alpha)$$

$(n-k+1)$- form

$$= \int_M \beta \wedge ((-1)^{(k-1)(n-k+1)} * * (-1)^k d * \alpha)$$

$$= \langle \beta, (-1)^{n(k+1)+1} * d * \alpha \rangle$$

$(k-1)(n-k+1) + k = n(k-1) - k^2 + k - 1 \equiv n(k+1) + 1$

Hence, $d^* = (-1)^{n(k+1)+1} * d *$

§ III Hodge theory : some key points

1° Denote $dd^* + d^*d$ by Δ, the Laplace-Beltrami operator

thm
- $\mathcal{H}^k = \{ \alpha \in \Omega^k(M) \mid \Delta \alpha = 0 \}$ is finite dimensional
- $\Omega^k(M) = \mathcal{H}^k \oplus d\Omega^{k-1} \oplus d^* \Omega^{k-1}$

$$= \mathcal{H}^k \oplus dd^* \Omega^k \oplus dd^* \Omega^k$$
$$= \mathcal{H}^k \oplus \Delta \Omega^k$$

Cor $H^k_{dR}(M) \cong H^{n-k}_{dR}(M)$

pf: $H^k_{dR} \cong \mathcal{H}^k \ni \alpha \iff d\alpha = 0 = * d * \alpha$

$\iff * d * (* \alpha) = 0 = d(* \alpha)$

$\iff * \alpha \in \mathcal{H}^{n-k}$

#

2° two key lemmas in the argument

① <u>thm</u> (compactness) $\{\alpha_n\}$: sequence in $\Omega^k(M)$. such that

$$|\alpha_n|_{L^2} \le C, \quad |\Delta\alpha_n|_{L^2} \le C$$

Then, $\{\alpha_n\}$ admits a Cauchy subsequence in $\Omega^k(M)$
(with respect to L^2-norm)

<u>rmk</u>
- no completeness / convergence statement
- $\mathcal{H}^k$ cannot be infinite dimensional :
  If so, $\exists \{\alpha_n\} \in \mathcal{H}^k$, L^2-orthonormal
  $\Rightarrow$ has no Cauchy subsequence. $\rightarrow\leftarrow$
- similar to last week
  $|\alpha|_{L^2_2} \le |\Delta\alpha|_{L^2} + |\alpha|_{L^2}$ and Sobolev

ⓘⓘ <u>thm</u> (regularity) $\alpha \in \Omega^k(M)$

If $\exists \beta \in L^2(M; \Lambda^k T^*M)$ square integrable k-form
such that $\langle \beta, \Delta\chi \rangle = \langle \alpha, \chi \rangle \quad \forall \chi \in \Omega^k(M)$,
then, β is smooth, and $\Delta\beta = \alpha$

<u>rmk</u> To prove

$$\Omega^k = \mathcal{H}^k \oplus \Delta\Omega^k$$
$$\eta = (\alpha \;,\; \eta - \alpha)$$

↖ L^2-orthogonal projection ($\dim \mathcal{H}^k < \infty$)

$\Rightarrow$ By functional analysis (Hahn-Banach ⋯)
$\exists \beta \in L^2 \ni \langle \beta, \Delta\chi \rangle = \langle \alpha, \chi \rangle$

§ IV. elliptic operator

1° $dd^* + d^*d = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ on $\mathbb{R}^n$

$= -g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} +$ (lower order terms)

ξ^i : parameter for Fourier transform

$(dd^* + d^*d)u = f$

$\Rightarrow -\sum_i (\xi^i)^2 \hat{u} = \hat{f} \Rightarrow \hat{u} = -\frac{1}{|\xi|^2} \hat{f}$

$\Rightarrow$ Fourier inversion solves u

only bad at $\xi = 0$

e.g. $\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ wave operator

$\rightsquigarrow \xi^2 - \eta^2 = (\xi+\eta)(\xi-\eta)$: singular (zero) along $\xi = \pm\eta$

2° In general, $D : \Gamma(E) \to \Gamma(F)$ k-th order differential operator

locally. $D = \sum_{|\alpha|=k} P^\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} +$ (lower order terms)

$P^\alpha(x) \in \mathrm{Hom}(E;F)$

defn ① The principal symbol is defined to be

$$\sigma_D(\xi) = \sum_{|\alpha|=k} P^\alpha(x) \xi^\alpha$$

← can formulate it as a tensor, which is confusing at a first glance. Let us not do it.

ⓘⓘ D is said to be elliptic,

if $\sigma_D(\xi)$ is invertible

for $\xi \neq 0$ (in particular, rank E = rank F)

e.g. • Laplace-Beltrami : $\Lambda^k T^*M \circlearrowleft$

$$\begin{bmatrix} -|\xi|^2 & & \\ & \ddots & \\ & & -|\xi|^2 \end{bmatrix} \quad \text{size} = \binom{n}{k} \times \binom{n}{k}$$

• Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

$\rightsquigarrow \frac{1}{2}(\xi_1 - i\xi_2)$

3° further remark ① $E_0, \cdots, E_n$: vector bundles over M

$D_j : \Gamma(E_j) \longrightarrow \Gamma(E_{j+1})$: 1st order differential operator

$$0 \longrightarrow \Gamma(E_0) \xrightarrow{D_0} \Gamma(E_1) \longrightarrow \cdots \xrightarrow{D_{j-1}} \Gamma(E_j) \xrightarrow{D_j} \cdots \longrightarrow \Gamma(E_n) \longrightarrow 0$$

is called an elliptic complex

if
- $D_j D_{j-1} = 0 \quad \forall j$
- $\forall \xi \neq 0$

$$0 \longrightarrow E_0 \xrightarrow{\sigma_{D_1}(\xi)} \cdots \xrightarrow{\sigma_{D_{j-1}}(\xi)} E_j \xrightarrow{\sigma_{D_j}(\xi)} \cdots \longrightarrow E_n \longrightarrow 0 \qquad (\star)$$

is a (long) exact sequence (over any $p \in M$)

② If so, $\sum_k \operatorname{rank} E_{2k} = \sum_k \operatorname{rank} E_{2k+1}$

Suppose that E_j carries an L^2-inner product

$$\Gamma(E_j) \underset{D_j^*}{\overset{D_j}{\rightleftarrows}} \Gamma(E_{j+1}) \quad \Rightarrow \quad \sigma_{D_j^*}(\xi) = (\sigma_{D_j}(\xi))^*$$

Consider $D_j^* D_j + D_{j-1} D_{j-1}^* \circlearrowright \Gamma(E_j)$

principal symbol $= (\sigma_{D_j})^* \sigma_{D_j} + \sigma_{D_{j-1}} \sigma_{D_{j-1}}^*$

For any $p \in M$ and $\xi \neq 0$, $(\star)$ is exact $\Leftrightarrow$ cohomology $= 0$

$\Leftrightarrow$ the above principal symbol is an isomorphism

③ Atiyah-Singer index theorem :

$$\sum_j (-1)^j \dim H^j(E, D) = \int \text{characteristic class of } (E, TM)$$

In general, only the Euler characteristic is topological, each $\dim H^j(E, D)$ may not be.

(de Rham is special : each cohomology is topological)