§I. tools from analysis

(Σ^2, g)

1° When $1 \le p < 2$. $L_1^p \hookrightarrow L^{2p/(2-p)}$

2° When $p > 2$. $L_1^p \hookrightarrow \mathcal{C}^{0,1-\frac{2}{p}}$

3° What about L_1^2?

By 1°, $L_1^2 \hookrightarrow L^q \quad \forall q \in [1, \infty)$, $\left(\begin{array}{c} |u|_{L^q} \le C(q) |u|_{L_1^2} \\ C(q) \xrightarrow[q\to\infty]{} \infty \end{array} \right)$

But $\exists\, u \in L_1^2$, $u \notin L^\infty = \mathcal{C}^0$

<u>prop</u> (Moser – Trudinger) $\exists\, C, \tilde{C} > 0$ such that

$$\int e^u \le C \exp(\tilde{C} |u|^2_{L_1^2}) \quad \forall u \in L_1^2$$

and $L_1^2 \to L^1$, $u \mapsto e^u$ is a compact map

(Let us assume this, and proceed)

4° $\Delta : \begin{array}{c} L^p_{k+2,0} \to L^p_{k,0} \\ \mathcal{C}_0^{k+2,\alpha} \to \mathcal{C}_0^{k,\alpha} \end{array}$ is an isomorphism

(on any reasonable space of functions.

kernel = {constant functions} = cokernel)

5° Jensen's inequality

$$\exp\left(\int u \Big/ \mathrm{Area}(\Sigma) \right) \le \int e^u \Big/ \mathrm{Area}(\Sigma)$$

(convexity: $\exp(tu + (1-t)v) \le t \exp(u) + (1-t)\exp(v)$)

§II. conformal change of Gaussian curvature

$g \rightsquigarrow \tilde{g} = e^{2u} g$.

$K \rightsquigarrow \tilde{K} = e^{-2u}(K - \Delta u)$ wrt g

1° $dV_g = \sqrt{\det g}\, dx\, dy \rightsquigarrow dV_{\tilde g} = \sqrt{\det \tilde g}\, dx\, dy = e^{2u}\, dV_g$

If $\Delta u = K - \tilde K e^{2u}$

$\Rightarrow \int \Delta u \, dV_g = 0 = \int K \, dV_g - \int \tilde K e^{2u} dV_g$

$\Rightarrow \int K \, dV_g = 2\pi \chi(\Sigma) = \int \tilde K \, dV_{\tilde g}$

$\Rightarrow$ The solution still satisfies the Gauss-Bonnet theorem.

2° genus $= 1$ (torus case) $\chi(\Sigma) = 2 - 2\cdot 1 = 0$

Given g on Σ, solve $\Delta u = K$ $(\tilde K = 0)$

By Gauss-Bonnet. $\int K \, dV_g = 0$

Hence, $\exists\ u \in C^{k,\alpha}$ $\forall\, k, \alpha$ with $\int u \, dV_g = 0$

satisfying $\Delta u = K$ ※

§III. higher genus case. $\chi(\Sigma) = 2 - 2\cdot \text{genus} < 0$

Solve u for $\Delta u = K + e^{2u}$ $(\tilde K = -1)$

1° uniqueness? $\Delta u = K + e^{2u}$, $\Delta v = K + e^{2v}$

$\Rightarrow \Delta(u-v) = e^{2u} - e^{2v} = e^{2v}(\underbrace{e^{2(u-v)} - 1}_{\text{same sign as } u-v})$

$\Rightarrow \int |\nabla(u-v)|^2 = -\int (u-v)\, \Delta(u-v)$

$= -\int e^{2v}(u-v)(e^{2(u-v)} - 1) \le 0$

Hence, $\nabla(u-v) \equiv 0 \Rightarrow v = u + C_0$

Back to the equation, $C_0 = 0$ ※

2° Solving $\Delta u = K + e^{2u}$ ← treat this as a constraint

turn this into a contant : easier to do estimate

3° By rescaling we may assume $\mathrm{Area}_g(\Sigma) = \int_\Sigma dV_g = -2\pi\,\chi(\Sigma)$

$\Rightarrow \int (K+1)\,dV_g = 0$

$\Rightarrow \exists!\, u_0$ with $\Delta u_0 = K+1$ and $\int u_0 = 0$ (u_0 = given datum)

Let $u = w + u_0$. $\Delta u = K + e^{2u}$

$\Leftrightarrow \Delta w + K + 1 = K + e^{2u_0} e^{2w}$

$\Leftrightarrow \Delta w + 1 = e^{2u_0} e^{2w}$

4° Consider $\mathcal{F}(w) = \int \frac{1}{2}|\nabla w|^2 - w$

$$\mathcal{F}(w + t\tilde{w}) = \int \cdots + t\big(\underbrace{\langle \nabla w, \nabla\tilde{w}\rangle - \tilde{w}}_{\text{by parts } `` (-\Delta w - 1)\tilde{w}}\big) + (t^2 \cdots)$$

$\Rightarrow \operatorname{grad}\mathcal{F} = -\Delta w - 1$

$S = \{ w \in L_1^2 \mid J(w) = \int e^{2u_0} e^{2w} = -2\pi\chi(\Sigma) \}$

$J(w + t\tilde{w}) = \int \cdots + 2t e^{2u_0} e^{2w}\tilde{w} + (t^2 \cdots)$

$\Rightarrow \operatorname{grad} J = 2 e^{2u_0} e^{2w}$

At the extremal of $\mathcal{F}$ over S, $\Delta w + 1 = \lambda e^{2u_0} e^{2w}$

$\int$ b.h.s $\Rightarrow \lambda = 1$.

5° existence of minimum?

By Jensen $\exp\left(\int \cancel{2u_0} + 2w \big/ c \right) \leq \int e^{2u_0 + 2w} = c$

$\Rightarrow \int w \leq c' \quad \forall w \in S$

Hence $\mathcal{F}(w) \geq -c'$ over S

$\mu = \inf_{w \in S} \mathcal{F}(w)$

6° $\exists\, w_j \in S$ such that $\mathcal{F}(w_j) = \int \frac{1}{2}|\nabla w_j|^2 - w_j \searrow \mu$

$|w_j|_{L_1^2}$: bounded, $|\nabla w_j|_{L^2}$: OKAY. Also, $-c' \leq -\int w_j \leq \mu + 1$

$\bar{w}_j = \int w_j \big/ \mathrm{Area}(\Sigma)$ (constant) $\Rightarrow |w_j|_{L^2} \leq |w_j - \bar{w}_j|_{L^2} + \underbrace{|\bar{w}_j|_{L^2}}_{\text{bounded}}$

$\wedge$ Poincaré

$\frac{1}{\sqrt{\lambda}} |\nabla(w_j - \cancel{\bar{w}_j})|_{L^2}$

7° $\{w_j\}$: bounded sequence in L^2_1

$$\Rightarrow \begin{cases} w_j \rightharpoonup w & \text{in } L^2_1 \\ w_j \to w & \text{in } L^2 \quad (\text{Sobolev}) \\ e^{2u_0} e^{2w_j} \to e^{2u_0} e^{2w} & \text{in } L^1 \quad (\text{Moser-Trudinger on } 2u_0 + 2w_j) \end{cases}$$

Hence, $w \in S = \{ w \in L^2_1 \mid J(w) = \int e^{2u_0} e^{2w} = -2\pi \chi(\Sigma) \}$

By a similar argument as that for the Poincaré inequality

$$\mathcal{F}(w) \le \mu = \inf \text{ of } \mathcal{F}$$

$\Rightarrow$ w achieves the minimum of $\mathcal{F}$ (over S)

8° no knowledge on $\nabla^2 w$ at moment

$$\int \underbrace{-\langle \nabla w, \nabla \varphi \rangle}_{= w \Delta \varphi} + \varphi - \lambda e^{2u_0} e^{2w} \varphi = 0 \qquad \forall \varphi \in C^\infty(\Sigma)$$

We say $\Delta w = -1 + \lambda e^{2u_0} e^{2w}$ weakly

$\exists !\, \lambda$ such that $\int_\Sigma \text{R.H.S.} = 0$

Sobolev $\Rightarrow$ $w \in L^2_1 \hookrightarrow L^2$

Moser–Trudinger on $4(u_0 + w)$ $\Rightarrow$ $e^{2u_0} e^{2w} \in L^2$

R.H.S $\in L^2_{0,0}$ $\Rightarrow$ $w \in L^2_{2,0}$ $\left(|u|_{L^2_2} \le C |\Delta u|_{L^2} \text{ and } C^\infty(\Sigma) \text{ is dense } \cdots \right)$

$\Rightarrow$ $w \in C^{1,\alpha}$ $\Rightarrow$ R.H.S $\in C^{1,\alpha}$

$\Rightarrow$ $w \in C^{3,\alpha}$

$\Rightarrow$ keep gaining regularity from the equation

(the bootstrapping argument)

$\Rightarrow$ $w \in C^\infty$ #