thm (Poincaré uniformization) Σ : compact without boundary, oriented
g: any metric. Then there exists $u \in C^\infty(M)$ such that $e^{2u} g$ has constant Gaussian curvature

discussion $\tilde{K} = e^{-2u}(-\Delta u + K)$ $\tilde{K}$: Gaussian curvature of $e^{2u} g$
K: Gaussian curvature of g

We have to solve $\Delta u = K - \tilde{K} e^{2u}$

$\Delta u = \operatorname{div}(\nabla u) = \operatorname{tr}(X \to \nabla_X(\nabla u)) = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g}\, g^{ij} \partial_j u)$

a scheme for solving geometric PDE

Consider the equation on some Lebesgue function space ⇝ Banach space
⇝ Easier to prove existence of solution
Normed vector space is easier to handle
{smooth functions} : Frechet space
⇝ Then prove the solution is indeed smooth

principle : Δ is an "elliptic" operator

§I. Sobolev space and Sobolev embedding

1° $f \in C^\infty(\Sigma)$. $L^p(\Sigma)$: completion of $C^\infty(M)$ by the norm
$|f|_{L^p} = (\int_\Sigma |f|^p)^{1/p}$ for $p \geq 1$

$L^p_1(\Sigma)$: use the norm
$|f|_{L^p_1} = (\int |f|^p)^{1/p} + (\int |\nabla f|^p)^{1/p}$ $\nabla f = g^{ij} \partial_j f \frac{\partial}{\partial x^i}$
$|\nabla f|^2 = g^{kl} \partial_k f\, \partial_l f$

One can similarly define $L^p_k(\Sigma)$ as well $\sum_{l=0}^{k} (\int |\nabla^{(l)} f|^p)^{1/p}$
They are Banach spaces (complete normed vector space)
When $p=2$, they are Hilbert spaces.

The Hölder space $C^{0,\alpha}(\Sigma)$ is defined to be

$$\mathcal{C}^{0,\alpha}(M) = \left\{ f : \text{functions on } \Sigma \;\middle|\; \sup_{\xi\neq\eta} \frac{|f(\xi)-f(\eta)|}{(d(\xi,\eta))^\alpha} < \infty \right\} \quad 0<\alpha\leq 1$$

Also a Banach space. one can check that $\mathcal{C}^1 \subset \mathcal{C}^{0,\alpha} \subset \mathcal{C}^0$

2° thm (Sobolev embedding I)

If $1\leq p<2$. $L^p_1(\Sigma) \hookrightarrow L^q(\Sigma)$ for any $q \leq \frac{2p}{2-p}$

If $q < \frac{2p}{2-p}$, the embedding is compact $\left(\frac{np}{n-p}\right)$

any $\{f_i\}$ with bounded L^p_1 norm, has convergent subsequence in L^q

pf: Only do the key case: functions on $\mathbb{R}^2$, supp $f \subset B(0;1)$

i) Start with $p=1$: $L^1_1 \hookrightarrow L^2$?

$$f(x,y) = \int_{-\infty}^x \partial f(s,y)\, ds$$

$$\Rightarrow |f(x,y)| \leq \int_{-\infty}^x |\partial_1 f(s,y)|\, ds \leq \int_{-\infty}^\infty |\partial_1 f(s,y)|\, ds$$

$$\Rightarrow |f(x,y)|^2 \leq \underbrace{\int_{-\infty}^\infty |\partial_1 f(s,y)|\, ds}_{S_1(y)} \cdot \underbrace{\int_{-\infty}^\infty |\partial_2 f(x,t)|\, dt}_{S_2(x)}$$

$$\int_{-\infty}^\infty |f(x,y)|^2\, dx \leq \int_{-\infty}^\infty S_1(y)\, S_2(x)\, dx = S_1(y) \int_{\mathbb{R}^2} |\partial_2 f(x,t)|\, dx\, dt$$

$\int_{-\infty}^\infty - \,dy$

$$= S_1(y)\, |\partial_2 f|_{L^1}$$

$$\Rightarrow \int_{-\infty}^\infty \int_{-\infty}^\infty |f|^2\, dx\, dy \leq |\partial_1 f|_{L^1}\, |\partial_2 f|_{L^1}$$

$$\Rightarrow |f|^2_{L^2} \leq |\nabla f|^2_{L^1} \leq |f|^2_{L^1_1}$$

ii) $1<p<2$ $\quad L^p_1 \hookrightarrow L^q \quad q \leq \frac{2p}{2-p}$

recall Hölder inequality: $\int uv \leq \left(\int u^p\right)^{\frac{1}{p}} \left(\int v^{p'}\right)^{\frac{1}{p'}} \quad \frac{1}{p}+\frac{1}{p'}=1$

$$\big|\,|f|^\gamma\big|_{L^2} \leq \big|\nabla |f|^\gamma\big|_{L^1} \leq \gamma \int |f|^{\gamma-1} |\nabla f|$$

$$\Rightarrow \left(\int |f|^{2\gamma}\right)^{\frac{1}{2}} \leq \gamma \left(\int |f|^{(\gamma-1)p'}\right)^{\frac{1}{p'}} \left(\int |\nabla f|^p\right)^{\frac{1}{p}}$$

$$2\gamma = (\gamma-1)p' \Rightarrow \frac{1}{p'} = \frac{\gamma-1}{2\gamma} = \frac{p-1}{p} \Rightarrow \gamma = \frac{p}{2-p}$$

$$\Rightarrow \left(\int |f|^{2p/2-p}\right)^{\frac{1}{2}} \le \frac{p}{2-p}\left(\int |f|^{2p/2-p}\right)^{p-1/p}\left(\int |\nabla f|^p\right)^{1/p}$$

$$\Rightarrow \left(\int |f|^{2p/2-p}\right)^{2-p/2p} \le \frac{p}{2-p}\left(\int |\nabla f|^p\right)^{\frac{1}{p}}$$

$$\Rightarrow \quad |f|_{L^{2p/2-p}} \le \frac{p}{2-p}|f|_{L^p_1}$$

※

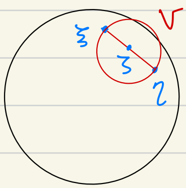
rmk
- Most text books uses Fourier transform formulation.
- We skip the compact embedding part
- For manifold, use partition of unity ...

3° thm (Sobolev embedding II)

When $p > 2$, $L^p_1(\Sigma) \hookrightarrow C^{0,\alpha}(\Sigma)$ for $0 < \alpha \le 1 - \frac{2}{p}$

If $\alpha < 1 - \frac{2}{p}$, the embedding is compact

pf: Again, do it over $\mathbb{R}^2$, and assume $\operatorname{supp} f \subset B(0;1)$



Fix any two ξ, η. let $\zeta = \xi + \eta/2$

$\rho = |\xi - \zeta| = |\eta - \zeta| = \frac{1}{2}|\xi - \eta|$

$V = B(\zeta, \rho)$

strategy : compare $f(\xi)$, $f(\eta)$ with $\frac{1}{\pi\rho^2}\iint_V f \, dx dy$

$$\forall z = (x,y) \in V, \quad f(z) - f(\xi) = f(\xi + t(z-\xi))\Big|_{t=0}^{1}$$

$$= \int_0^1 \frac{d}{dt} f(\xi + t(z-\xi)) \, dt$$

$$= \int_0^1 \langle \nabla f|_{\xi + t(z-\xi)}, z - \xi \rangle \, dt$$

Integrate over $z \in V$

$$\Rightarrow \int_V f(z)\, dx dy - \pi\rho^2 f(\xi) = \int_V \int_0^1 \langle \nabla f|_{\xi + t(z-\xi)}, z - \xi \rangle \, dt \, dx dy$$

$$\Rightarrow \left|\int_V f(z)\, dx dy - \pi\rho^2 f(\xi)\right| \le 2\rho \int_V \int_0^1 \left|\nabla f|_{\xi + t(z-\xi)}\right| \, dt \, dx dy$$

estimate by switching the order of integration

∇f in $\tilde{z} = \xi + t(z-\xi)$ $\quad \tilde{x} = tx + \boxed{t,\xi}$ $\quad \tilde{y} = ty + \boxed{t,\xi}$

$\Rightarrow dx\,dy = t^{-2}\, d\tilde{x}\, d\tilde{y}$

$|z-\xi| < \rho \Rightarrow |z-\xi| < 2\rho \Rightarrow |\tilde{z}-\xi| < 2t\rho$

$z \in V \Rightarrow \tilde{z} \in B(\xi, 2t\rho)$ (contains more region)

$$\Rightarrow \int_V \int_0^1 \left| \nabla f|_{\xi + t(z-\xi)} \right| dt\, dx\, dy \le \int_0^1 t^{-2} \Big(\int_{B(\xi, 2t\rho)} |\nabla f(\tilde{z})|\, d\tilde{x}\, d\tilde{y} \Big) dt$$

$$\int_{B(\xi,2t\rho)} |\nabla f(\tilde{z})|\, d\tilde{x}\, d\tilde{y} \le \Big(\int 1^{p'}\Big)^{\frac{1}{p'}} \Big(\int |\nabla f(\tilde{z})|^p\Big)^{\frac{1}{p}} \qquad \frac{1}{p'} + \frac{1}{p} = 1$$

$$\le (\pi 4t^2\rho^2)^{\frac{1}{p'}} |\nabla f|_{L^p} \rightsquigarrow \text{on supp}(f)$$

$$\Rightarrow \int_V \int_0^1 \left| \nabla f|_{\xi + t(z-\xi)} \right| dt\, dx\, dy \le |\nabla f|_{L^p} (4\pi\rho^2)^{\frac{1}{p'}} \int_0^1 t^{-2+\frac{2}{p'}}\, dt \qquad -2+\frac{2}{p'} = -\frac{2}{p} \quad (p>2)$$

$$= C(p)\, |\nabla f|_{L^p}\, \rho^{1+\frac{2}{p'}}$$

$$\Rightarrow \left| f(\xi) - \frac{1}{\pi\rho^2} \int_V f(z)\, dx\, dy \right| \le \pi^{-1} C(p)\, |\nabla f|_{L^p}\, \rho^{-1+\frac{2}{p'}} \qquad -1+\frac{2}{p'} = 1-\frac{2}{p} \qquad \text{—}(*)$$

- $\int |f| \le \left(\int 1^{p'}\right)^{\frac{1}{p'}} \left(\int |f|^p\right)^{\frac{1}{p}} \Rightarrow |f|_{L^1} \le C' |f|_{L^p}$

  By taking $\rho = 1$ in $(*)$, $\sup f \le C'' |f|_{L^p_1}$

- $(*)$ holds for η as well. By triangle inequality,

  $$|f(\xi) - f(\eta)| \le \underset{= \xi - \eta/2}{\rho}^{\,1-\frac{2}{p}}\, C''' |\nabla f|_{L^p}$$

Hence, $|f|_{C^{0,\alpha}} \le \tilde{C} |f|_{L^p_1}$ ✗

§ II. linear theory of Laplacian

0° Given f, solve $\Delta u = f$?

If so, $g^{ij} u_{;ij} = u^{;i}{}_{;i} = f \Rightarrow \int \Delta u = 0 = \int f$

At least, $\int f$ must vanish.

($\Delta u = 1$ has no solution on $\mathcal{C}^\infty(\Sigma)$)

Indeed, $\underbrace{\int v \Delta u + \langle \nabla v, \nabla u \rangle}_{= \operatorname{div}(v \nabla u)} = 0$

$\Rightarrow \int \langle u, \Delta u \rangle = -\int |\nabla u|^2$

I° prop (Poincaré inequality) $\exists \lambda > 0$ such that

$$\int u^2 \leq \frac{1}{\lambda} \int |\nabla u|^2$$

in a way, λ is the 1^{st} non zero eigenvalue of $-\Delta$

$$\forall u \in L^2_{1,0} = \{ u \in L^2_1 , \int u = 0 \}$$

rmk It fails for $u = $ Const. and $\int u = 0$ is a necessary condition

pf: fact $L^2_{1,0}$ is a Hilbert space (complete, inner product space)

Let $\lambda = \inf \{ |\nabla u|^2 \mid u \in L^2_{1,0} , \int u^2 = 1 \}$

We have to show $\lambda > 0$.

Suppose that $\{v_i\}$ is a minimizing sequence in $L^2_{1,0}$

$$\int v_i^2 = 1 , \quad \int |\nabla v_i|^2 \to \lambda$$

- Then, $\{v_i\}$ is bounded in $L^2_1 \subset L^{3/2}_1 \overset{\text{compact}}{\hookrightarrow} L^2$ by Sobolev

  $\Rightarrow$ We may assume $\{v_i\}$ converges in L^2

- fact any bounded sequence has a weak convergent subsequence (in a Hilbert space)

  $\Rightarrow \{v_i\} \rightharpoonup v$ in L^2_1

Namely, $\forall w \in L^2_1$, $\langle w, v_i \rangle_{L^2_1} = \int \langle w, v_i \rangle + \langle \nabla w, \nabla v_i \rangle$

pointwise multiplication

pointwise inner product

$\downarrow i \to \infty$

$\langle w, v \rangle$

i) Since $v_i \to v$ in L^2, $\int (v_i - v) \le C \cdot \int (v_i - v)^2 \to 0$

$$\Rightarrow \int (v_i - v) = -\int v = 0 \Rightarrow v \in L^2_{1,0}$$

ii) $\underbrace{|v_i|_{L^2}}_{=1} - \underbrace{|v - v_i|_{L^2}}_{\to 0} \le |v|_{L^2} \le \underbrace{|v_i|_{L^2}}_{=1} + \underbrace{|v - v_i|_{L^2}}_{\to 0} \Rightarrow \int v^2 = 1$

iii) $\underbrace{|v - v_i|^2_{L^2}}_{\to 0} = \underbrace{|v|^2_{L^2}}_{=1} - 2\langle v, v_i\rangle_{L^2} + \underbrace{\langle v_i, v_i\rangle_{L^2}}_{=1} \Rightarrow \langle v, v_i\rangle_{L^2} \to 1$

iv) (use $w = v$ in $v_i \rightharpoonup v$) $\underbrace{\langle v_i, v\rangle_{L^2_1}}_{= \langle v_i, v\rangle_{L^2} + \int \langle \nabla v_i, \nabla v\rangle} \longrightarrow \underbrace{\langle v, v\rangle_{L^2_1}}_{= \langle v, v\rangle_{L^2} + \int |\nabla v|^2}$

iii) & iv)

$$\Rightarrow \int |\nabla v|^2 = \lim_{i\to\infty} \int \langle \nabla v_i, \nabla v\rangle \le \liminf_i \left(\int |\nabla v_i|^2\right)^{\frac{1}{2}} \left(\int |\nabla v|^2\right)^{\frac{1}{2}}$$

$$\le \lambda^{\frac{1}{2}} \left(\int |\nabla v|^2\right)^{\frac{1}{2}}$$

$\Rightarrow \int |\nabla v|^2 \le \lambda$. By the definition of λ $\Rightarrow \int |\nabla v|^2 = \lambda$

If $\lambda = 0 \Rightarrow \nabla v = 0 \Rightarrow v =$ constant. but $\int v = 0 \Rightarrow v = 0$

$\rightarrow\leftarrow$ with $\int v^2 = 1$ #

natural, since Δ: 2^{nd} order differentiation

2° thm $\Delta : L^p_{k+2,0}(\Sigma) \to L^p_{k,0}(\Sigma)$ is an isomorphism

$\Delta(\text{const}) = 0$ — if $\int f \neq 0$, $\Delta u = f$ has no solution

Const $= \ker \Delta$ ($\Delta u = 0 \Rightarrow 0 = -\int u \Delta u = \int |\nabla u|^2$)

Key of the proof: $p = 2$, $k = 0$ $\Delta : L^2_{2,0} \to L^2_{0,0}$

$\exists\, C > 0$ such that $|u|_{L^2_2} \le C\, |\Delta u|_{L^2}$ ——— (✡)

($\Rightarrow$ injectivity of Δ) $\forall\, u \in L^2_2$, $\int u = 0$

$\Delta u = f$, $\int u = 0 \Rightarrow |\nabla^2 u|_{L^2} + |\nabla u|_{L^2} + |u|_{L^2} \leq C |f|_{L^2}$?

i) By Poincaré, $|u|^2_{L^2} \leq \frac{1}{\lambda} |\nabla u|^2_{L^2}$

ii) $|\nabla u|^2_{L^2} = \int |\nabla u|^2 = -\int u \Delta u = -\int uf \leq \frac{\varepsilon}{2} \int u^2 + \frac{1}{2\varepsilon} \int f^2$

$\Rightarrow |\nabla u|^2_{L^2} \leq \frac{\varepsilon}{2\lambda} |\nabla u|^2_{L^2} + \frac{1}{2\varepsilon} |f|^2_{L^2}$ choose $\varepsilon = \lambda$

$\Rightarrow |\nabla u|^2_{L^2} \leq C(\lambda) |f|^2_{L^2}$

So far, $|\nabla u|_{L^2} + |u|_{L^2} \leq C' |f|_{L^2}$

$|\nabla^2 u|^2_{L^2} = g^{ik} g^{jl} u_{;kl} u_{;ij} = u^{;ij} u_{;ij}$

$\operatorname{div}(u^{;ij} u_{;i}) = u^{;ij}{}_{j} u_{;i} + u^{;ij} u_{;ij}$

$\Rightarrow \int |\nabla^2 u|^2 = -\int u^{;ij}{}_{j} u_{;i}$

$u^{;ij}{}_{j} = u^{;ji}{}_{j} = g^{ki} \underset{\nabla^2_{j;k}}{u^{;j}{}_{kj}} = g^{ki} \underset{\nabla^2_{k;j}}{u^{;j}{}_{jk}} - g^{ki} R^{j}{}_{lkj} u^{;l}$

$= g^{ki} f_{;k} - g^{ki} R^{j}{}_{lkj} u^{;l}$

$-\int u^{;ij}{}_{j} u_{;i} = \underbrace{-\int g^{ki} f_{;k} u_{;i}} \underbrace{- \int g^{ki} R^{j}{}_{lkj} u^{;l} u_{;i}}$

$-\int \langle \nabla f, \nabla u \rangle = \int f \Delta u = \int f^2$

$- g^{ki} R^{j}{}_{lkj} u^{;l} u_{;i} = R^{j}{}_{ljk} u^{;l} u^{;k} = Ric(\nabla u, \nabla u)$

$\Rightarrow \int |\nabla^2 u|^2 = \int f^2 + \int Ric(\nabla u, \nabla u) \leq \int f^2 + c \int |\nabla u|^2 \leq C' \int f^2$

surjectivity of Δ ?

$f \in L^2$ $\int f = 0$ Solve $\Delta u = f$?

Consider $\mathcal{D}_f(u) = \int \frac{1}{2}|\nabla u|^2 + f u$ $\quad -\int v \Delta u$

$\mathcal{D}_f(u + t v) = \int \frac{1}{2}|\nabla u|^2 + f u + t\left(\langle \nabla u, \nabla v\rangle + f v\right) + (\cdots t^2 \cdots)$

u : critical of $\mathcal{D}_f \iff \Delta u = f$

Consider $\mathcal{D}_f(u)$ on $L^2_{1,0}$

- it is convex : $\mathcal{D}_f((1-s)u + s\tilde{u}) \geq (1-s)\mathcal{D}_f(u) + s\,\mathcal{D}_f(\tilde{u})$
- $\mathcal{D}_f(u) \geq \int \frac{1}{2}|\nabla u|^2 - \frac{2}{\lambda} f^2 - \frac{\lambda}{8} u^2$

$\geq \int \frac{1}{2}|\nabla u|^2 + \frac{\lambda}{8} u^2 \underbrace{- \frac{\lambda}{4} u^2}_{\geq -\frac{1}{4}|\nabla u|^2 \text{ by Poincaré}} - \frac{2}{\lambda} f^2$

$\geq c_0 |u|_{L^2_1} - c_1$

$\mathcal{D}_f(u)$ admits a minimum on $L^1_{1,0}$

$\Rightarrow$ "weak" solution of $\Delta u = f$

$\left(\begin{array}{l} \int v(\Delta u) = -\int \langle \nabla v, \nabla u \rangle = \int (\Delta v) u \\ \text{weak solution: } \int u(\Delta h) - h f = 0 \quad \forall h \in \mathcal{C}^\infty(\Sigma) \end{array} \right)$

By $(\star)$ and $\mathcal{C}^\infty(\Sigma)$ is dense $\cdots \Rightarrow \cdots$ $u \in L^2_{1,0}$ $\cdots$ #