§I. tensor calculus

1° For $V = v^i \frac{\partial}{\partial x^i}$. write $\nabla_{\frac{\partial}{\partial x^j}} V$ as $V^i{}_{;j} \frac{\partial}{\partial x^i}$

where $V^i{}_{;j} = \partial_j V^i + \Gamma^i_{jk} V^k$

2° $\nabla^2_{k,l} V = (\nabla\nabla V)(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}) = (\nabla_{\frac{\partial}{\partial x^k}} \nabla V)(\frac{\partial}{\partial x^l})$

$= \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^l}} V - \nabla_{\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l}} V$

$V^i{}_{;lk} \frac{\partial}{\partial x^i} = \nabla_{\frac{\partial}{\partial x^k}} (V^i{}_{;l} \frac{\partial}{\partial x^i}) - \Gamma^j_{kl} V^i{}_{;j} \frac{\partial}{\partial x^i}$

$= (\partial_k (V^i{}_{;l}) + \Gamma^i_{kj} V^j{}_{;l} - \Gamma^j_{kl} V^i{}_{;j}) \frac{\partial}{\partial x^i}$

3° a tensor of type (p,q) is a smooth section of $(\otimes^p TM) \otimes (\otimes^q T^*M)$

$B = B^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}$

$\rightsquigarrow$ components of $\nabla_{\frac{\partial}{\partial x^k}} B$ is denoted by $B^{i_1 \cdots i_p}{}_{j_1 \cdots j_q ; k}$, which is

$\partial_k (B^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}) + \sum_{\mu=1}^{p} \Gamma^{i_\mu}_{kl} B^{i_1 \cdots \hat{i}_\mu l \cdots i_p}{}_{j_1 \cdots j_q} - \sum_{\nu=1}^{q} \Gamma^{l}_{k j_\nu} B^{i_1 \cdots i_p}{}_{j_1 \cdots \hat{j}_\nu l \cdots j_q}$

4° For a 1-form $\alpha = \alpha_i dx^i$

$\nabla_k \alpha = \alpha_{i;k} dx^i$, $\alpha_{i;k} = \partial_k \alpha_i - \Gamma^l_{ik} \alpha_l$

dual vector field $= \tilde{\alpha}^i \frac{\partial}{\partial x^i}$, $\tilde{\alpha}^i = g^{ij} \alpha_j$ $\quad$ g^{ij} : inverse of g

$\tilde{\alpha}^i{}_{;k} = \partial_k \tilde{\alpha}^i + \Gamma^i_{km} \tilde{\alpha}^m$

$= \partial_k (g^{ij} \alpha_j) + \Gamma^i_{km} g^{ml} \alpha_l$

$= g^{ij} (\partial_k \alpha_j - (\partial_k g_{jm}) g^{ml} \alpha_l + \frac{1}{2} (\partial_k g_{jm} + \partial_m g_{jk} - \partial_j g_{mk}) g^{ml} \alpha_l)$

$= g^{ij} (\partial_k \alpha_j - \Gamma^l_{jk} \alpha_l) = g^{ij} \alpha_{j;k}$

This is the coordinate-calculation proof of $\nabla g = 0$

5° curvature tensor $R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}) \frac{\partial}{\partial x^j} = R^i{}_{jkl} \frac{\partial}{\partial x^i}$

$R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{km} \Gamma^m_{jl} - \Gamma^i_{lm} \Gamma^m_{jk}$

$$(\nabla_k(\nabla V))(\partial_\ell) = \nabla_k \nabla_\ell V - \nabla_{\nabla_k \ell} V = V^i{}_{;\ell k} \frac{\partial}{\partial x^i} \quad (\text{also } V^i{}_{;\ell;k})$$

$\shortparallel$

recall $\nabla^2_{k,\ell} V - \nabla^2_{\ell,k} V = R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}) V$

$\Rightarrow \quad V^i{}_{;\ell k} - V^i{}_{;k\ell} = R^i{}_{jk\ell} V^j$

Similarly $\quad \alpha_{i;\ell k} - \alpha_{i;k\ell}$ $\qquad \tilde{\alpha}^j = g^{jm} \alpha_m$

$= g_{ij} (\tilde{\alpha}^j{}_{;\ell k} - \tilde{\alpha}^j{}_{;k\ell})$

$= g_{ij} R^j{}_{mk\ell} \tilde{\alpha}^m = R_{imk\ell} g^{mj} \alpha_j = R_i{}^j{}_{k\ell} \alpha_j = -R^j{}_{ik\ell} \alpha_j$

<u>recall</u> $\quad R_{ijk\ell} = -R_{jik\ell}$ $\qquad \langle R(\partial_k, \partial_\ell) \partial_j, \partial_i \rangle = \langle R(\partial_k, \partial_\ell) \partial_i, \partial_j \rangle$

$\Rightarrow g^{im} R_{ijk\ell} = -g^{jm} R_{jik\ell}$

$\Rightarrow R_i{}^m{}_{k\ell} = -R^m{}_{ik\ell}$

Also, for $\quad a = a_{ij}\, dx^i \otimes dx^j$

$a_{ij;\ell k} - a_{ij;k\ell} = -R^m{}_{ik\ell} a_{mj} - R^m{}_{jk\ell} a_{im}$

(similar formula for any (p,q)-type tensor)

6° divergence : recall $\boxed{\text{final}}$

$V \in \mathfrak{X}(M)$, $\quad \operatorname{div} V = \operatorname{tr}(X \mapsto \nabla_X V) \in C^\infty(M)$

$V = V^i \frac{\partial}{\partial x^i} \Rightarrow \nabla V = V^i{}_{;j}\, dx^j \otimes \frac{\partial}{\partial x^i} \Rightarrow \operatorname{div} V = V^j{}_{;j}$

$\frac{\partial}{\partial x^j} \mapsto V^i{}_{;j} \frac{\partial}{\partial x^i}$

$\operatorname{div}(V)\, d\mathrm{vol} = d(\iota(V)\, d\mathrm{vol})$

<u>prop</u> For any $V \in \mathfrak{X}(M)$ with compact support

$$\int_M \operatorname{div}(V)\, d\mathrm{vol} = 0 \quad \Big(= \int_{\text{“}\partial M\text{”}} \iota(V)\, d\mathrm{vol} \Big)$$

§ II. variation of scalar curvature.

0° $\operatorname{Ricci}(U,V) = \sum_i \langle R(e_i, U) V, e_i \rangle = \operatorname{tr}(X \mapsto R(X,U)V)$

$\operatorname{Ricci} = R^i{}_{ki\ell}\, dx^k \otimes dx^\ell$

$= g^{ij} R_{jki\ell}\, dx^k \otimes dx^\ell \in \Gamma(\operatorname{Sym}^2 T^* M)$

We can take trace (with respect to g) to obtain the "scalar curvature"

$$s = \sum_{i,j} \langle R(e_i, e_j) e_j, e_i \rangle$$
$$= g^{ij} g^{kl} R_{ikjl} = g^{kl} R^i{}_{kil}$$

$$\left(S = S_{kl}\, dx^k \otimes dx^l \xrightarrow{\text{dual}} S_{kl}\, g^{lm}\, dx^k \otimes \frac{\partial}{\partial x^m} \longrightarrow \text{trace} = S_{kl}\, g^{lk} \right)$$

1° $\mathcal{H}(g) = \int_M s(g)\, dvol_g$ (Hilbert functional)

<u>question</u> One way to find "best" metric : look for critical state of $\mathcal{H}(g)$ $\rightsquigarrow$ calculate its variation

$$g(t) = g_{ij}(t)\, dx^i \otimes dx^j \rightsquigarrow \left.\frac{d}{dt}\right|_{t=0} g_{ij} = a_{ij} \; : \text{symmetric } (0,2)\text{-tensor}$$

$$\Gamma^i_{jk}(t) \rightsquigarrow \left.\frac{d}{dt}\right|_{t=0} \Gamma^i_{jk}(t) \text{ is a } (1,2)\text{-tensor}$$

$$\left(\nabla^t - \nabla^0 \text{ is } C^\infty(M)\text{-linear} \in \Gamma(\mathrm{Hom}(TM \otimes TM ; TM)) \right)$$

2° $$g^{ij} g_{jk} = \delta^i_k \Rightarrow \left(\left.\frac{d}{dt}\right|_{t=0} g^{ij} \right) g_{jk} + g^{ij} a_{jk} = 0$$
$$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} g^{ij} = - g^{il} a_{lm} g^{mj}$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} \Gamma^i_{jk} = -\frac{1}{2} g^{im} a_{mn} g^{nl} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$
$$+ \frac{1}{2} g^{il} (\partial_j a_{lk} + \partial_k a_{jl} - \partial_l a_{jk})$$
$$= \frac{1}{2} g^{il} (\partial_j a_{lk} + \partial_k a_{jl} - \partial_l a_{jk}) - g^{im} \Gamma^n_{kj} a_{mn}$$
$$= \frac{1}{2} g^{il} (\partial_j a_{lk} + \partial_k a_{jl} - \partial_l a_{jk} - 2 \Gamma^m_{kj} a_{lm})$$

$$+ \quad a_{lk;j} = \partial_j a_{lk} - \Gamma^m_{jl} a_{mk} - \Gamma^m_{jk} a_{lm}$$
$$+ \quad a_{jl;k} = \partial_k a_{jl} - \Gamma^m_{kj} a_{ml} - \Gamma^m_{jl} a_{jm}$$
$$- \quad a_{jk;l} = -\partial_l a_{jk} + \Gamma^m_{lj} a_{mk} + \Gamma^m_{lk} a_{jm}$$

Hence, $\frac{d}{dt}\Big|_{t=0} \Gamma^i_{jk} = \frac{1}{2} g^{il} (a_{lk;j} + a_{jl;k} - a_{jk;l})$

3° $R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{km} \Gamma^m_{jl} - \Gamma^i_{lm} \Gamma^m_{jk}$

At first, write $\frac{d}{dt}\Big|_{t=0} \Gamma^i_{jk} = B^i_{jk}$

$\Rightarrow \frac{d}{dt}\Big|_{t=0} R^i{}_{jkl} = \partial_k B^i_{lj} - \partial_l B^i_{kj} + B^i_{km} \Gamma^m_{jl} + \Gamma^i_{km} B^m_{jl} - B^i_{lm} \Gamma^m_{jk} - \Gamma^i_{lm} B^m_{jk}$

$$B^i_{lj;k} = \partial_k B^i_{lj} + \Gamma^i_{km} B^m_{lj} - \cancel{\Gamma^m_{kl} B^i_{mj}} - \Gamma^m_{kj} B^i_{lm}$$

$$- B^i_{kj;l} = -\partial_l B^i_{kj} \underset{-}{+} \Gamma^i_{lm} B^m_{kj} \underset{+}{-} \cancel{\Gamma^m_{lk} B^i_{mj}} \underset{+}{-} \Gamma^m_{lj} B^i_{km}$$

$\Rightarrow \frac{d}{dt}\Big|_{t=0} R^i{}_{jkl} = B^i_{lj;k} - B^i_{kj;l}$

((Then, plug in $B^i_{jk} = \frac{1}{2} (a^i{}_{k;j} + a_j{}^i{}_{;k} - a_{jk}{}^{;i})$

$\frac{1}{2} (a^i{}_{j;lk} + a_l{}^i{}_{;jk} - a_{lj}{}^{;i}{}_k - a^i{}_{j;kl} - a_k{}^i{}_{;jl} + a_{kj}{}^{;i}{}_l)$

4° $s = g^{jl} R^i{}_{jil}$

$\frac{d}{dt}\Big|_{t=0} s = - g^{jk} a_{km} g^{ml} R^i{}_{jil}$

$+ \frac{1}{2} \underbrace{g^{jl} (a^i{}_{j;li} + a_l{}^i{}_{;ji} - a_{lj}{}^{;i}{}_i - a^i{}_{j;il} - a_i{}^i{}_{;jl} + a_{ij}{}^{;i}{}_l)}$

$(a^{ij}{}_{;j} + a^{ji}{}_{;j} - a^j{}_j{}^{;i} - a^{ji}{}_{;j} - a_j{}^j{}_;{}^i + a_j{}^{i;j})_{;i}$

$= \operatorname{div}(V_a)$

$dvol = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n$

$\frac{d}{dt}\Big|_{t=0} dvol = \frac{1}{2} (\det g)^{-\frac{1}{2}} (\det g)\, g^{ij} a_{ji}\, dx^1 \wedge \cdots \wedge dx^n = \frac{1}{2} g^{ij} a_{ij}\, dvol$

$\Rightarrow \frac{d}{dt}\Big|_{t=0} s\, dvol = \left(- g^{jk} a_{km} g^{ml} R^i{}_{jil} + \frac{1}{2} g^{ij} a_{ij}\, s + \operatorname{div}(V_a) \right) dvol$

If g is a critical point of the Hilbert functional,

$$0 = \int_M \left(- g^{jk} a_{km} g^{ml} R^i{}_{jil} + \frac{s}{2} g^{ij} a_{ij} + \operatorname{div}(V_a) \right) dvol$$

$\forall$ symmetric $(0,2)$-tensor a_{ij}

By considering any a_{ij} of compact support,

$$g^{jk} g^{ml} R^{i}{}_{jil} - \frac{s}{2} g^{km} = 0$$

$$g_{k\mu} g_{m\nu} \quad \delta^{j}_{\mu} \delta^{l}_{\nu} R^{i}{}_{jil} - \frac{s}{2} \delta^{m}_{\mu} g_{m\nu} = 0$$

$$\Rightarrow R^{i}{}_{\mu i \nu} - \frac{s}{2} g_{\mu\nu} = 0$$

$$\text{Ricci} = R^{i}{}_{\mu i \nu} dx^{\mu} \otimes dx^{\nu} = \frac{s}{2} g_{\mu\nu} dx^{\mu} \otimes dx^{\nu}$$

rmk $R^{i}{}_{jil} g^{jk} g^{ml} a_{km} = \langle \text{Ricci}, a \rangle$

inner product on $T^*M \otimes T^*M$ induced by g

5° further discussion

If $R^{i}{}_{\mu i \nu} = \frac{s}{2} g_{\mu\nu}$. taking trace on μ, ν gives

$$\sum_{\mu,\nu,i} g^{\mu\nu} R^{i}{}_{\mu i \nu} = \frac{s}{2} \sum_{\mu,\nu} g_{\mu\nu} g^{\nu\mu} \Rightarrow s = \frac{s}{2} \dim M$$

When $\dim M > 2 \Rightarrow s = 0 \Rightarrow \text{Ricci} = 0$

We can consider the "constraint" variation: fixing the total volume

$$V(g) = \int_M d\mathrm{vol}_g$$

$$\left.\frac{d}{dt}\right|_{t=0} V(g) = \int_M \frac{1}{2} \langle g, a \rangle \, d\mathrm{vol} \quad \Rightarrow \text{"}\nabla V\text{"} = \frac{1}{2} g$$

We know "$\nabla \mathcal{H}$" $= -\text{Ricci} + \frac{s}{2} g$

By Lagrange multiplier, critical of $\mathcal{H}$ on $V =$ constant happens at $\nabla \mathcal{H} = \lambda \nabla V$

$$\Rightarrow \text{Ricci} - \frac{s}{2} g - \frac{\lambda}{2} g = 0$$

Ricci // g : Einstein metric

rmk This is a formal calculation. It does not guarantee the existence of critical points

§III. Laplace operator

1° $f \longmapsto df = \partial_i f \, dx^i$

Dirichlet energy $\mathcal{D}(f) = \int |df|^2 \, d\mathrm{vol}$

$$|df|^2 = g^{ij} \partial_i f \, \partial_j f$$

f : one parameter family of functions, $\left.\frac{d}{dt}\right|_{t=0} f = h$

$$\left.\frac{d}{dt}\right|_{t=0} g^{ij} \partial_i f \, \partial_j f = 2 g^{ij} \partial_i f \, \partial_j h$$

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{D}(f) = 2 \int g^{ij} \partial_i f \, \partial_j h \sqrt{\det g} \, dx^1 \wedge \cdots \wedge dx^n$$

$$= -2 \int \partial_j \left(g^{ij} \sqrt{\det g} \, \partial_i f \right) h \frac{1}{\sqrt{\det g}} \, d\mathrm{vol}$$

$$\text{“}\nabla \mathcal{D}(f)\text{”} = -2 \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^j} \left(g^{ij} \sqrt{\det g} \, \partial_i f \right)$$

2° $$\frac{\partial}{\partial x^j}\left(g^{ij} \sqrt{\det g} \right) = -g^{ik} \underbrace{(\partial_j g_{kl}) g^{lj}}_{= (\partial_l g_{kj}) g^{jl}} \sqrt{\det g} + \frac{1}{2} \sqrt{\det g} \, g^{i\overset{k}{\cancel{j}}} g^{\overset{j}{\cancel{k}}l} \partial_{\underset{k}{\cancel{j}}} g_{\underset{j}{\cancel{k}}l}$$

$$= -\sqrt{\det g} \, \frac{1}{2} g^{lj} g^{ki} \left(\partial_j g_{kl} + \partial_l g_{jk} - \partial_k g_{jl} \right)$$

$$= -\sqrt{\det g} \, g^{lj} \Gamma^i_{jl}$$

$$\Rightarrow -2 \nabla \mathcal{D}(f) = g^{ij} \partial_j \partial_i f - g^{\overset{i}{\cancel{l}}j} \Gamma^{\overset{k}{\cancel{i}}}_{j\underset{i}{\cancel{l}}} \partial_{\underset{k}{\cancel{i}}} f$$

$$= g^{ij} \left(\partial_j (\partial_i f) - \Gamma^k_{ji} \partial_k f \right) = g^{ij} (\partial_i f)_{;j}$$

3° $$\operatorname{div}(\nabla f) = \operatorname{div}\left(g^{ij} \partial_i f \frac{\partial}{\partial x^j} \right)$$

$$= \left(g^{ij} \partial_i f \right)_{;j} = g^{ij} (\partial_i f)_{;j}$$