calculating the Levi-Civita connection and Riemann curvature tensor

0° $\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$

$R^k_{lij} = \partial_i \Gamma^k_{jl} - \partial_j \Gamma^k_{il} + \cdots$ usually a little bit complicated

$\{e_i\}$: local orthonormal frame for $T\Sigma$

$\nabla e_i = \omega_i^j \otimes e_j$ How to calculate ω_i^j ?

let $\{\omega^i\}$: dual orthonormal coframe (for $T^*\Sigma$)

$\nabla \omega^i = -\omega^i_j \otimes \omega^j$

1° $\delta_{ij} = \langle e_i, e_j \rangle \Rightarrow 0 = \langle \nabla e_i, e_j \rangle + \langle e_i, \nabla e_j \rangle$

$0 = \omega_i^j + \omega_j^i$

2° $\nabla_{e_i} e_j - \nabla_{e_j} e_i = [e_i, e_j]$

$[e_i, e_j] = T^k_{ij} e_k \Rightarrow T^k_{ij} = \omega^k([e_i, e_j])$

<u>recall</u> $(d\alpha)(U,V) = U(\alpha(V)) - V(\alpha(U)) - \alpha([U,V])$

$\Rightarrow \omega^k([e_i, e_j]) = e_i(\cancel{\delta^k_j})^0 - e_j(\cancel{\delta^k_i})^0 - (d\omega^k)(e_i, e_j)$

$= \omega^k(\nabla_{e_i} e_j - \nabla_{e_j} e_i)$ torsion free

Hence, $\omega_j^k(e_i) - \omega_i^k(e_j) = -(d\omega^k)(e_i, e_j)$

<u>rmk</u> This is equivelent to $d\omega^k = -\omega^k_l \wedge \omega^l$

(plug in (e_i, e_j) on both sides)

3° $\oplus$ $\omega_j^k(e_i) - \omega_i^k(e_j) = -(d\omega^k)(e_i, e_j)$

$\ominus$ $\omega_i^j(e_k) - \omega_k^j(e_i) = -(d\omega^j)(e_k, e_i)$

$\ominus$ $\omega_k^i(e_j) - \omega_j^i(e_k) = -(d\omega^i)(e_j, e_k)$

$\Rightarrow 2\omega_j^i(e_k) = (d\omega^i)(e_j, e_k) + (d\omega^j)(e_k, e_i) - (d\omega^k)(e_i, e_j)$

<u>Upshot</u> the Levi-Civita connection can be found by calculating $\{d\omega^i\}$'s

This is nothing more than proving the existence and uniqueness of the Levi-Civita connection in this terminology.

4° curvature tensor $F = dA + A \wedge A$

$$= d\omega^i_j + \omega^i_k \wedge \omega^k_j = \langle R(-,-)e_j, e_i \rangle$$

<u>example</u> S^n : unit sphere in $\mathbb{R}^{n+1}$

Use stereographic projection $(x^1, \dots, x^n) \longmapsto (2x^1, \dots, 2x^n, 1-|x|^2) / 1+|x|^2$

$$\Rightarrow g = \sum_{j=1}^{n} \frac{4}{(1+|x|^2)^2} (dx^j)^2$$

$\omega^j = \frac{2}{1+|x|^2} dx^j$, $e_j = \frac{1+|x|^2}{2} \frac{\partial}{\partial x^j}$ orthonormal (co) frame

$$d\omega^j = - \frac{4}{(1+|x|^2)^2} x^k dx^k \wedge dx^j$$

$$= + x^k \omega^j \wedge \omega^k = - \omega^j_k \wedge \omega^k$$

$$\Rightarrow \omega^j_k = - x^k \omega^j + x^j \omega^k$$

$\omega^j_k = -\omega^k_j$ and solves as well

$$d\omega^i_j + \omega^i_k \wedge \omega^k_j = d(x^i \omega^j - x^j \omega^i) + (x^i \omega^k - x^k \omega^i) \wedge (x^k \omega^j - x^j \omega^k)$$

$$= (1 + \cancel{|x|^2}) \omega^i \wedge \omega^j + \cancel{x^i x^k \omega^j \wedge \omega^k} - \cancel{x^j x^k \omega^i \wedge \omega^k}$$

$$+ \cancel{x^i x^k \omega^k \wedge \omega^j} + \cancel{x^k x^j \omega^i \wedge \omega^k} - \cancel{(x^k)^2} \omega^i \wedge \omega^j$$

$$\Rightarrow \langle R(-,-)e_j, e_i \rangle = \omega^i \wedge \omega^j$$

$\langle R(e_i, e_j) e_j, e_i \rangle = 1$ others are zero (except equivalent ones)