<u>recall</u> $\Sigma \subset \mathbb{R}^3$ regular surface

$$g_{ij} = \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle$$

$$h_{ij} = \left\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, N \right\rangle = -\left\langle \frac{\partial X}{\partial x^i}, \frac{\partial N}{\partial x^j} \right\rangle$$

$K = \det(h_{ij}) / \det(g_{ij})$: Gaussian curvature

Theorema Egregium, $K = K(g_{ij}, \partial g_{ij}, \partial^2 g_{ij})$

$\boxed{\text{Question}}$ What we have done so far is sort of backward.

$$g_{ij} \rightsquigarrow R_{ijkl}$$

relation to the change of "normal"?

§I. basic notions.

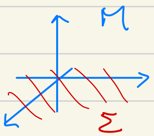
<u>setting</u> $(M^n, \bar g) \supset \Sigma^k$: submanifold $(k < n)$

last semester, $(M, \bar g) = (\mathbb{R}^3, dx^2 + dy^2 + dz^2)$

$\Rightarrow g = \bar g|_\Sigma$ is a Riemannian metric on Σ

1° <u>preparation</u> $U, V \in \mathfrak{X}(\Sigma)$ Since $M \supset \Sigma \cong \mathbb{R}^n \supset \mathbb{R}^k \times \{0\}$

locally, $\exists\, \bar U, \bar V \in \mathfrak{X}(M)$, which extend U, V



$\mathbb{R}^k \times \mathbb{R}^{n-k}$
$(x, \quad y)$

$$U = u^i(x) \frac{\partial}{\partial x^i}$$

$$\bar U = \bar u^i(x,y) \frac{\partial}{\partial x^i} + \bar u^\alpha(x,y) \frac{\partial}{\partial y^\alpha} \qquad \bar u^i(x,0) = u^i(x), \quad \bar u^\alpha(x,0) \equiv 0$$

$$[\bar U, \bar V] = \left[\bar u^i(x,y) \frac{\partial}{\partial x^i} + \bar u^\alpha(x,y)\frac{\partial}{\partial y^\alpha},\ \bar V^j(x,y)\frac{\partial}{\partial x^j} + \bar V^\beta(x,y)\frac{\partial}{\partial y^\beta}\right]$$

$$= \bar u^i \frac{\partial \bar V^j}{\partial x^i}\frac{\partial}{\partial x^j} + \bar u^i \frac{\partial \cancelto{0}{\bar V^\beta}}{\partial x^i}\frac{\partial}{\partial y^\beta} + \cancel{\bar u^\alpha}\frac{\partial \bar V^j}{\partial y^\alpha}\frac{\partial}{\partial x^j} + \cancel{\bar u^\alpha}\frac{\partial \bar V^\beta}{\partial y^\alpha}\frac{\partial}{\partial y^\beta} - \cdots$$

$$\Rightarrow [\bar U, \bar V]\big|_{(x,0)} = u^i \frac{\partial V^j}{\partial x^i}\frac{\partial}{\partial x^j} - V^i \frac{\partial u^j}{\partial x^i}\frac{\partial}{\partial x^j} = [U, V]$$

Hence, $[\bar U, \bar V]\big|_\Sigma = [U, V]$

2° <u>lemma</u> $\nabla_U V = \left(\bar\nabla_U \bar V\big|_\Sigma\right)^{T\Sigma}$, the Levi-Civita connection of g

over Σ, $U = \bar U$ is given by "projection" pf: To see $(\bar\nabla_U \bar V)^{T\Sigma}$ is a connection and is independent of the extension $\bar V$, use $M \supset \Sigma \cong \mathbb{R}^n \supset \mathbb{R}^k \times \{0\}$, and do similar calculation as in 1°.

$$U(\langle V, W\rangle_g) = \bar U(\langle \bar V, \bar W\rangle_{\bar g})|_\Sigma$$
$$= \langle \bar\nabla_{\bar U}\bar V, \bar W\rangle|_\Sigma + \langle \bar V, \bar\nabla_{\bar U}\bar W\rangle|_\Sigma$$
$$= \langle (\bar\nabla_U \bar V)^{T\Sigma}, W\rangle + \langle V, (\bar\nabla_U \bar W)^{T\Sigma}\rangle$$

Also, $[U, V] = [\bar U, \bar V]|_\Sigma = (\bar\nabla_{\bar U}\bar V - \bar\nabla_{\bar V}\bar U)|_\Sigma$ ⇝ hence in $T\Sigma$
$$= (\bar\nabla_U \bar V)^{T\Sigma} - (\bar\nabla_V \bar U)^{T\Sigma}$$

By the uniqueness of the Levi-Civita connection. DONE #

3° What has been dropped?

(from now on, write U for $\bar U$ for simplicity)

$\bar\nabla_U V - \nabla_U V$ is normal to Σ, $(\bar\nabla_U V)^{N\Sigma}$

Note that $\bar\nabla_U(fV) - \nabla_U(fV) = f(\bar\nabla_U V - \nabla_U V)$

Hence, the second fundamental form is a tensor

$$\mathrm{II}: T\Sigma \times T\Sigma \to N\Sigma \quad \text{normal bundle of } \Sigma$$
$$(U, V) \mapsto (\bar\nabla_U V)^{N\Sigma} \quad (T\Sigma)^\perp \text{ in } TM|_\Sigma$$

recall that in regular surfaces theory, II is symmetric $\left(\frac{\partial^2 X}{\partial x^i \partial x^j} = \frac{\partial^2 X}{\partial x^j \partial x^i}\right)$

<u>lemma</u> II is symmetric

pf: $\mathrm{II}(U, V) = (\bar\nabla_U V)^{N\Sigma}$

But $\bar\nabla_U V - \bar\nabla_V U = [U, V]$: tangent

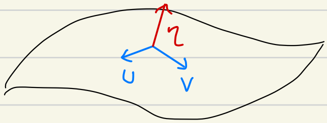
$\Rightarrow (\bar\nabla_U V)^{N\Sigma} - (\bar\nabla_V U)^{N\Sigma} = 0$ #

<u>rmk</u> This measures the rate change of normal as well

η: local section of $N\Sigma$

$\langle \mathrm{II}(U,V), \eta \rangle = \langle (\bar\nabla_U V)^{N\Sigma}, \eta \rangle = \langle \bar\nabla_U V, \eta \rangle$ since $\langle V, \eta \rangle \equiv 0$

$= -\langle V, \bar\nabla_U \eta \rangle$



Σ $\sim$ the V-component of the rate of change of η in the U-direction.

4° What about the normal components of the rate of change of N?

$$\Gamma(N\Sigma) \longrightarrow \Gamma(T^*\Sigma \otimes N\Sigma)$$
$$\eta \longmapsto (\bar\nabla \eta)^{N\Sigma}$$

denote in by $\nabla^\perp \eta$

NOT function linear in η.

This is nothing more than a connection on $N\Sigma$

rmk This is not interesting in the case of regular surfaces in $\mathbb{R}^3$

η : unit normal of Σ (up to ± 1, unique)

$\Rightarrow \nabla^\perp \eta \,/\!/\, \eta$, $\langle \nabla^\perp \eta, \eta \rangle = \langle \bar\nabla \eta, \eta \rangle = \frac{1}{2} d\langle \eta, \eta \rangle = 0$

5° example demonstration :

claim. $\mathrm{II}(U,V) = (\bar\nabla_U \bar V)^{N\Sigma}$ is independent of the choice of the extension $\bar V$

We will show that $\mathrm{II}(U,V)|_p = 0$ if $V(p) = 0$

Let $\{e_j\}_{j=1}^k$: (local) trivializing sections for $T\Sigma$

$\{N_\alpha\}_{\alpha=k+1}^n$: (local) trivializing sections for $N\Sigma$

$\rightsquigarrow$ choose a smooth extension $\{\bar e_j, \bar N_\alpha\}$: trivializing sections for TM

$\Rightarrow V = v^j e_j \qquad v^j \in \mathcal{C}^\infty(\Sigma),\ v^j(p) = 0$

$\Rightarrow \bar V = \bar v^j \bar e_j + \bar v^\alpha N_\alpha$. where $\bar v^j, \bar v^\alpha \in \mathcal{C}^\infty(M)$

with $\bar v^j|_\Sigma = v^j$. $\bar v^\alpha|_\Sigma \equiv 0$

$$\bar\nabla_U \bar V = \bar\nabla_U(\bar v^j \bar e_j + \bar v^\alpha \bar N_\alpha)$$
$$= \underbrace{U(\bar v^j)\bar e_j}_{T\Sigma} + \underbrace{\bar v^j(\bar\nabla_U \bar e_j)}_{\text{vanish at } p} + \underbrace{U(\bar v^\alpha)}_{=0}\bar N_\alpha + \underbrace{\bar v^\alpha}_{=0}(\bar\nabla_U \bar N_\alpha)$$

along Σ

$\Rightarrow (\bar\nabla_U \bar V|_p)^{N\Sigma} = 0$ #

§ II. fundamental equations

$$\bar{R}(U,V)W = \bar{\nabla}_U\bar{\nabla}_V W - \bar{\nabla}_V\bar{\nabla}_U W - \bar{\nabla}_{[U,V]}W$$

1° Suppose that $U, V, X, Y \in T_p\Sigma$.

relation between $\langle \bar{R}(U,V)X, Y\rangle$ and $\langle R(U,V)X, Y\rangle$?

$$\bar{\nabla}_U\bar{\nabla}_V X = \bar{\nabla}_U(\nabla_V X + \mathrm{II}(V,X))$$

always use § I, 1° & 2°

$$\Rightarrow \langle \bar{\nabla}_U\bar{\nabla}_V X, Y\rangle = \langle \bar{\nabla}_U \nabla_V X, Y\rangle + \langle \bar{\nabla}_U(\mathrm{II}(V,X)), Y\rangle$$

since $Y \in T_p\Sigma$

$$= \langle \nabla_U \nabla_V X, Y\rangle - \langle \mathrm{II}(V,X), \mathrm{II}(U,Y)\rangle$$

$$\langle \bar{\nabla}_V\bar{\nabla}_U X, Y\rangle = \langle \nabla_V \nabla_U X, Y\rangle - \langle \mathrm{II}(U,X), \mathrm{II}(V,Y)\rangle$$

$$\langle \bar{\nabla}_{[U,V]}X, Y\rangle = \langle \nabla_{[U,V]}X, Y\rangle$$

<u>thm</u> (Gauss equation)

$$\langle \bar{R}(U,V)X, Y\rangle = \langle R(U,V)X, Y\rangle - \langle \mathrm{II}(V,X), \mathrm{II}(U,Y)\rangle + \langle \mathrm{II}(U,X), \mathrm{II}(V,Y)\rangle$$

2° Suppose that $U, V, W \in T_p\Sigma$, what is the normal component of $\bar{R}(U,V)W$? Take any $\eta \in N_p\Sigma$

(extend it as a local section, for computational purpose)

Denote $\langle \mathrm{II}(-,-), \eta\rangle$ by $\mathrm{II}(-,-;\eta)$

the second fundamental form in the direction of η

$$\bar{\nabla}_U\bar{\nabla}_V W = \bar{\nabla}_U(\nabla_V W + \mathrm{II}(V,W))$$

$$\langle \bar{\nabla}_U\bar{\nabla}_V W, \eta\rangle = \langle \bar{\nabla}_U \nabla_V W, \eta\rangle + \langle \bar{\nabla}_U \mathrm{II}(V,W), \eta\rangle$$

$$= \mathrm{II}(U, \nabla_V W; \eta) + \langle \nabla_U^\perp \mathrm{II}(V,W), \eta\rangle$$

$$= \mathrm{II}(U, \nabla_V W; \eta) + U(\mathrm{II}(V,W;\eta)) - \mathrm{II}(V,W; \nabla_U^\perp \eta)$$

$$\langle \bar{\nabla}_V\bar{\nabla}_U W, \eta\rangle = \mathrm{II}(V, \nabla_U W; \eta) + V(\mathrm{II}(U,W;\eta)) - \mathrm{II}(U,W;\nabla_V^\perp \eta)$$

$$\langle \bar{\nabla}_{[U,V]}W, \eta\rangle = \mathrm{II}([U,V], W; \eta)$$

$$= \mathrm{II}(\nabla_U V - \nabla_V U, W; \eta)$$

$$\Rightarrow \langle \bar{R}(U,V)W, \eta\rangle = \underline{\mathrm{II}(U, \nabla_V W; \eta)} + U(\mathrm{II}(V,W;\eta)) - \mathrm{II}(V,W;\nabla_U^{\perp}\eta)$$
$$- \mathrm{II}(V, \nabla_U W; \eta) - \underline{V(\mathrm{II}(U,W;\eta))} + \underline{\mathrm{II}(U,W;\nabla_V^{\perp}\eta)}$$
$$- \mathrm{II}(\nabla_U V, W; \eta) + \underline{\mathrm{II}(\nabla_V U, W; \eta)}$$

Regard $\mathrm{II}(-,-;-)$ as a section of $T^*\Sigma \otimes T^*\Sigma \otimes (N\Sigma)^*$

$\rightsquigarrow$ $(T\Sigma, \nabla)$ and $(N\Sigma, \nabla^{\perp})$ induce a metric connection, $\nabla^{\perp}$.

by requiring the Leibniz rule — use same notation

Namely, $(\nabla_V^{\perp}\mathrm{II})(U,W;\eta) = V(\mathrm{II}(U,W;\eta))$
$$- \mathrm{II}(\nabla_V U, W;\eta) - \mathrm{II}(U, \nabla_V W;\eta) - \mathrm{II}(U,W;\nabla_V^{\perp}\eta)$$

$\underline{\cdots} = -(\nabla_V^{\perp}\mathrm{II})(U,W;\eta)$ $\qquad$ $\underline{\cdots} = (\nabla_U^{\perp}\mathrm{II})(V,W;\eta)$

thm (Codazzi equation)
$$\langle \bar{R}(U,V)W, \eta\rangle = (\nabla_U^{\perp}\mathrm{II})(V,W;\eta) - (\nabla_V^{\perp}\mathrm{II})(U,W;\eta)$$

3° $\nabla^{\perp}$: metric connection on $N\Sigma \to \Sigma$

What is its curvature?

$$R^{\perp}(U,V)\eta = \nabla_U^{\perp}\nabla_V^{\perp}\eta - \nabla_V^{\perp}\nabla_U^{\perp}\eta - \nabla^{\perp}_{[U,V]}\eta$$

Try to compare it with $(\bar{R}(U,V)\eta)^{N\Sigma}$

Tangential part of $\bar{\nabla}_V\eta$: $\langle \bar{\nabla}_V \eta, U\rangle = -\mathrm{II}(U,V;\eta)$

Denote $(\bar{\nabla}_V\eta)^{T\Sigma}$ by $-S_\eta(V)$ $\qquad$ $= -\langle S_\eta(V), U\rangle$

Namely, $S_\eta \in \mathrm{Hom}(T\Sigma)$ indeed, symmetric

$$\bar{\nabla}_U\bar{\nabla}_V\eta = \bar{\nabla}_U(-S_\eta(V) + \nabla_V\eta)$$
$$= -\nabla_U(S_\eta(V)) - \mathrm{II}(S_\eta(V), U) + (\bar{\nabla}_U(\nabla_V^{\perp}\eta))^{T\Sigma} + \nabla_U^{\perp}\nabla_V^{\perp}\eta$$

$$\Rightarrow \langle \bar{\nabla}_U\bar{\nabla}_V\eta, \xi\rangle = -\langle S_\eta(V), S_\xi(U)\rangle + \langle \nabla_U^{\perp}\nabla_V^{\perp}\eta, \xi\rangle$$

Also, $\langle \bar{\nabla}_V\bar{\nabla}_U\eta, \xi\rangle = -\langle S_\eta(U), S_\xi(V)\rangle + \langle \nabla_V^{\perp}\nabla_U^{\perp}\eta, \xi\rangle$

Clearly, $\langle \bar{\nabla}_{[U,V]}\eta, \xi\rangle = \langle \nabla^{\perp}_{[U,V]}\eta, \xi\rangle$

thm (Ricci equation)

$$\langle \bar{R}(U,V)\eta, \xi \rangle = \langle R^{\perp}(U,V)\eta, \xi \rangle + \langle [S_\xi, S_\eta]U, V \rangle$$

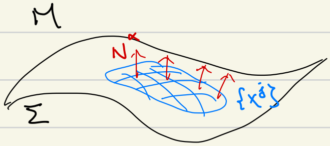
rmk Again, when codim $\Sigma = 1$, each term is zero.

the theorem is empty

§ III. coordinate expression

0° $\{x^i\}$: local coordinate on Σ $\rightsquigarrow$ $\{\frac{\partial}{\partial x^i}\} \in \mathfrak{X}(\Sigma)$

$\{N_\alpha\}$: local orthonormal, trivializing sections of $N\Sigma$



$$h^\alpha_{ij} := \langle \mathrm{II}(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}), N_\alpha \rangle$$

$$\Rightarrow \mathrm{II} = h^\alpha_{ij}(dx^i \otimes dx^j) \otimes N_\alpha \qquad h^\alpha_{ij} = h^\alpha_{ji}$$

$$\overset{S_{N_\alpha}}{\overset{\shortparallel}{}}\langle S_\alpha(\tfrac{\partial}{\partial x^i}), \tfrac{\partial}{\partial x^j} \rangle = h^\alpha_{ij} \Rightarrow S_\alpha = g^{jk} h^\alpha_{ki}\, dx^i \otimes \frac{\partial}{\partial x^j}$$

1° Gauss eqn : $\bar{R}_{k\ell ij} = \langle \bar{R}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^k} \rangle$

$= R_{k\ell ij} - h^\alpha_{j\ell} h^\alpha_{ik} + h^\alpha_{jk} h^\alpha_{i\ell}$ (orthonormal frame $\rightsquigarrow$ mess up upper-lower)

2° Codazzi eqn : $\nabla^\perp N_\alpha = A^\beta_\alpha N_\beta$ $\qquad A^\beta_\alpha \in \Omega^1(\Sigma)$

$\nabla \frac{\partial}{\partial x^i} = \Gamma^k_{ij}\, dx^j \otimes \frac{\partial}{\partial x^k}$

<u>Convention</u> • $\eta = \eta^\alpha N_\alpha$

$$\nabla^\perp_{\frac{\partial}{\partial x^i}} \eta = \left(\frac{\partial}{\partial x^i}\eta^\alpha + A^\alpha_\beta\left(\frac{\partial}{\partial x^i}\right)\eta^\beta \right) N_\alpha =: \eta^\alpha_{\ ;i} N_\alpha$$

• $U = U^j \frac{\partial}{\partial x^j}$

$$\nabla_{\frac{\partial}{\partial x^i}} U = \left(\frac{\partial}{\partial x^i} U^j + \Gamma^j_{ik} U^k \right) \frac{\partial}{\partial x^j} =: U^j_{\ ;i} \frac{\partial}{\partial x^j}$$

• $$\left(\nabla^\perp_{\frac{\partial}{\partial x^i}} \mathrm{II}\right)\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}; N_\alpha\right) = \frac{\partial}{\partial x^i}\left(\mathrm{II}\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}; N_\alpha\right)\right) - \mathrm{II}\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}; N_\alpha\right)$$
$$- \mathrm{II}\left(\frac{\partial}{\partial x^j}, \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}; N_\alpha\right) - \mathrm{II}\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}; \nabla^\perp_{\frac{\partial}{\partial x^i}} N_\alpha\right)$$
$$= \frac{\partial}{\partial x^i} h^\alpha_{jk} - h^\alpha_{\ell k}\Gamma^\ell_{ij} - h^\alpha_{j\ell}\Gamma^\ell_{ik} + h^\beta_{jk} A^\alpha_\beta\left(\frac{\partial}{\partial x^i}\right) =: h^\alpha_{jk;i}$$

$$\mathrm{II} = h^\alpha_{ij}\, dx^i \otimes dx^j \otimes N_\alpha$$

• $$\bar{R}_{\alpha k i j} = \left\langle \bar{R}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}, N_\alpha \right\rangle$$
$$= h^\alpha_{jk;i} - h^\alpha_{ik;j}$$

3° Ricci eqn : $$R_{\beta\alpha ij} = \left\langle \bar{R}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) N_\alpha, N_\beta \right\rangle$$
$$= F^\beta_{\alpha\, ij} + g^{\ell k}\left(h^\beta_{j\ell} h^\alpha_{ik} - h^\alpha_{j\ell} h^\beta_{ik} \right)$$

4° a <u>theorem</u> of Bonnet. $\Sigma^2 \subset \mathbb{R}^3$ only one normal direction

$$\left.\begin{aligned} 0 &= R_{1212} - h_{11}h_{22} + h_{12}^2 \\ 0 &= h_{21;1} - h_{12;2} \\ 0 &= h_{22;1} - h_{12;2} \end{aligned}\right\} \star$$

$i \neq j, k$

<u>thm</u> $\forall\ g_{ij} > 0,\ h_{ij}$ satisfy $\star$

$\Rightarrow$ locally $\exists$ regular surface with these data