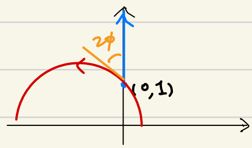
§I. Gauss lemma (cont'd)

1° <u>example</u> $\left(\mathbb{H},\ \frac{1}{y^2}(dx^2+dy^2)\right)$



$\gamma(t) = (0, e^t)$ geodesic with $\begin{cases} \gamma(0) = (0,1) \\ \gamma'(0) = (0,1) \end{cases}$

isometry preserves geodesic

$$z \longmapsto \frac{\cos\phi\, z + \sin\phi}{-\sin\phi\, z + \cos\phi} \quad : \text{fixing } (0,1) = i$$

$$\rightsquigarrow \frac{\cos\phi\, e^r i + \sin\phi}{-\sin\phi\, e^r i + \cos\phi} \quad \text{is a geodesic from } (0,1)$$

(x,y)-components:

$$\gamma_\theta(r) = \left(\frac{\sin\theta \sinh r}{\cosh r - \cos\theta \sinh r},\ \frac{1}{\cosh r - \cos\theta\sinh r}\right) \qquad \theta = 2\phi$$

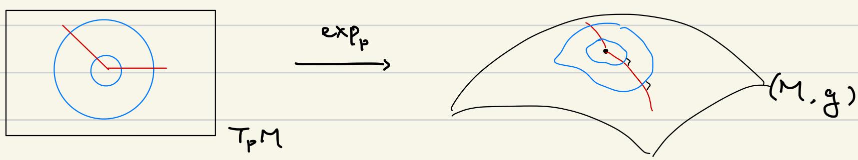
This is exactly the exponential map : $\begin{cases} \gamma_\theta(0) = 0 \\ \gamma_\theta'(0) = (-\sin\theta, \cos\theta) \end{cases}$

(r,θ) : polar coordinate for $\mathbb{R}^2 = T_i\mathbb{H} \longrightarrow \gamma_\theta(r) \in \mathbb{H}$

In terms of this parametrization, $\frac{1}{y^2}(dx^2+dy^2) = dr^2 + \sinh^2 r\, d\theta^2$

(direct calculation)

2° Gauss lemma



"spherical" coordinate $\mathbb{R}_+ \times S^{n-1} \longrightarrow$ end point of geodesic
$r,\ (\theta^1, \dots, \theta^{n-1})$
with length r, initial velocity in the direction of $(\theta^1, \dots, \theta^n)$

Then. $g = dr^2 + \sum_{j,k=1}^{n-1} \tilde{g}_{jk}(r,\theta)\, d\theta^j \otimes d\theta^k$

coefficient in dr is 1. and no $dr \otimes d\theta^j$ component.

3° $\mathbb{R}^2$: $dr^2 + r^2 d\theta^2$ 　　　　 Taylor at $r=0$

S^2 : $dr^2 + \sin^2 r \, d\theta^2$ 　　　 $\sin^2 r = r^2 - \frac{1}{3} r^4 + \cdots$

$\mathbb{H}$: $dr^2 + \sinh^2 r \, d\theta$ 　　　 $\sinh^2 r = r^2 + \frac{1}{3} r^4 + \cdots$

rmk / advertisement • $\underbrace{r^2 + 0 \cdot r^3}_{\text{smoothness of } g} - \frac{1}{3} K(p) r^4$

- $K > 0$ ⇝ geodesics at p shall "close up"
- $K < 0$ ⇝ geodesic sphere at p shall become "larger"

(The key tool : second variational formula of $E[\gamma]$ / $L[\gamma]$)

§II. Hopf-Rinow

Question How large of t can $\exp_p(tv)$ be defined ?

(example in class)

thm (Hopf-Rinow) The following statements are equivalent

ⓘ (M, d) is a complete metric space

ⓘⓘ For some $p \in M$, $\exp_p$ is defined on all T_pM

ⓘⓘⓘ For any $q \in M$, $\exp_q$ is defined on all T_qM

Any of them imply

ⓘⓥ $\forall p, q$, $\exists$ geodesic $\gamma \in \Omega_{p,q}$ $\ni$ $L[\gamma] = d(p,q)$

(for open ball in $\mathbb{R}^n$, ⓘⓥ is true, but ⓘ is NOT)

- Let us only do this claim : ⓘⓘ$_p$ $\Rightarrow$ $\forall q \in M$, $\exists \gamma \in \Omega_{p,q}$ geodesic $\ni L[\gamma] = d(p,q)$

1° If $q \in \exp_p(\overline{B(0;\varepsilon)})$ in prop*, DONE.

Suppose not. claim' $\exists q' \in \partial \overline{B}$ $\ni$ $d(p,q) = \varepsilon + d(q',q)$

pf: $\gamma(t) \in \Omega_{p,q}$, $\exists t_1 \ni \gamma(t_1) \in \partial \overline{B}$

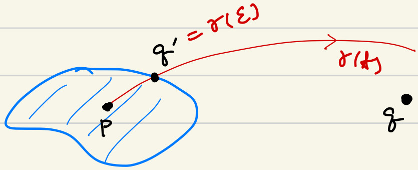
$$\Rightarrow L[\gamma] \underset{\text{By prop*}}{\geq} \varepsilon + L[\gamma|_{[t_1,1]}] \geq \varepsilon + d(\partial \overline{B}, q)$$

Hence, $d(p,q) \geq \varepsilon + d(\partial \bar{B}, q)$. "$\leq$" by Δ-inequality

$\Rightarrow d(p,q) = \varepsilon + d(\partial \bar{B}, q)$.

Since $\partial \bar{B}$ is compact, $\exists\, q' \in \partial \bar{B}$ achiving $d(\partial \bar{B}, q)$ *

2°



$q' = \exp_p(\varepsilon v)$ for some $v \in T_pM$, $|v| = 1$

Let $\gamma(t) = \exp_p(tv)$

<u>assumption</u> $\gamma(t)$: defined $\forall\, t \in \mathbb{R}$

closed

Consider $I = \{ t \in [0, d(p,q)] \mid d(p, \gamma(t)) + d(\gamma(t), q) = d(p,q) \} \ni 0, \varepsilon$

Let $T = \max I$

3° If $T = d(p,q)$, $d(p,q) = d(p, \gamma(T)) + d(\gamma(T), q) \Rightarrow d(\gamma(T), q) = 0$

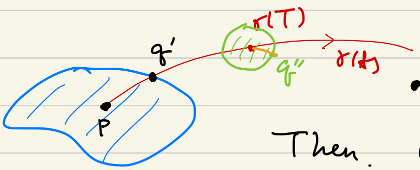
$\overset{\geq}{L[\gamma|_{[0,T]}]} \geq d(p,q) \leftarrow$ and equality holds.

4° If $T < d(p,q)$, apply 1° on $\gamma(T)$, q

call it σ

$\Rightarrow \exists\, q''$: endpoint of a short geodesic at $\gamma(T)$

$\rightarrow d(\gamma(T), q'') + d(q'', q) = d(\gamma(T), q)$



Then $d(p, \gamma(T)) + d(\gamma(T), q'') + d(q'', q) = d(p,q)$

$\Rightarrow d(p, \gamma(T)) + d(\gamma(T), q'') = d(p,q) - d(q'', q) \geq d(p, q'')$

Hence, $d(p, \gamma(T)) + d(\gamma(T), q'') = d(p, q'')$

$\Rightarrow d(p, q'')$ is achieved by $\gamma|_{[0,T]} \cup \sigma$: no broken

$\Rightarrow \gamma'(T) = -\sigma'(0) \Rightarrow T$ is NOT max $\rightarrow\leftarrow$

- Finish (ii) $\Rightarrow$ (i)

Suppose that $\{q_i\}$ is a Cauchy sequence in (M, d)

By the claim, $\exists$ geodesics at p of unit speed γ_i

such that $q_i = \gamma_i(T_i)$ and $d(p, q_i) = T_i$

$d(q_i, q_j) \geq d(p, q_i) - d(p, q_j) = T_i - T_j$

Hence, $\{T_i\}$ is Cauchy

$\gamma_i(t) = \exp_p(t v_i)$ for $v_i \in T_pM$ of unit length

By compactness of $S^{n-1} \subset T_pM$ and completeness of $\mathbb{R}$

(after passing to a subsequence) $v_i \to v$ $T_i \to T$

By (ii), $q = \exp_p(Tv)$ is defined

With the continuity of $\exp_p$, $\lim q_j = \lim \exp_p(T_j v_j)$

$= \exp_p(\lim T_j v_j) = q$

$\divideontimes$

§ III. Jacobi field

goal: Suppose that $\gamma_s(t)$ is a family of geodesics, what can be said about $V(t) = \frac{\partial\gamma}{\partial s}\Big|_{s=0}$?

0° local calculation. $\{x^i\}$: coordinate for M

$\{\Gamma^k_{ij}(x)\}$: Christoffel symbols of Levi-Civita connection

$\gamma_s(t)$: $\{x^i = x^i(s,t)\}$

$\tilde{\nabla}$: pull-back of the Levi-Civita connection on $\gamma^*TM \to [0,1]\times(-\varepsilon,\varepsilon)$

$$\left(\tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial x^i} = \Gamma^k_{ij}\frac{\partial x^j}{\partial t}\frac{\partial}{\partial x^k} \quad \text{and} \quad \tilde{\nabla}_{\frac{\partial}{\partial s}}\frac{\partial}{\partial x^i} = \Gamma^k_{ij}\frac{\partial x^j}{\partial s}\frac{\partial}{\partial x^k}\right)$$

1° $\gamma_s(t)$ is a geodesic $\iff \tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial\gamma}{\partial t} = 0$ $= \tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial x^k}{\partial t}\frac{\partial}{\partial x^k}$

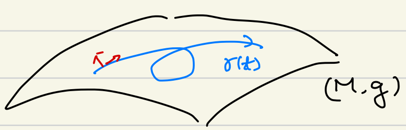
$$= \left(\frac{\partial^2 x^k}{\partial t^2} + \Gamma^k_{ij}\frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial t}\right)\frac{\partial}{\partial x^k}$$

Consider
$$\begin{aligned}
\tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial\gamma}{\partial s} &= \tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{\frac{\partial}{\partial s}}\frac{\partial\gamma}{\partial t} \\
&= \tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{\frac{\partial}{\partial s}}\frac{\partial\gamma}{\partial t} - \tilde{\nabla}_{\frac{\partial}{\partial s}}\underbrace{\tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial\gamma}{\partial t}}_{=0} - \tilde{\nabla}_{[\frac{\partial}{\partial t},\frac{\partial}{\partial s}]}\frac{\partial\gamma}{\partial t} \\
&= F_{\tilde{\nabla}}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial s}\right)\frac{\partial\gamma}{\partial t} \quad \text{pull-back of the curvature of } \nabla \text{ by } \gamma \\
&= R\left(\frac{\partial\gamma}{\partial t},\frac{\partial\gamma}{\partial s}\right)\frac{\partial\gamma}{\partial t}
\end{aligned}$$

If $V(t) = \frac{\partial\gamma}{\partial s}\Big|_{s=0}$ is the variational field of geodesics, it must be a Jacobi field, i.e. it satisfies $\nabla_{\frac{\partial\gamma}{\partial t}}\nabla_{\frac{\partial\gamma}{\partial t}}V = R\left(\frac{\partial\gamma}{\partial t}, V\right)\frac{\partial\gamma}{\partial t}$

Riemann curvature tensor

2° As a second order ODE system



$\gamma(t)$: geodesic (assume $|\gamma'(t)| = 1$)

Write $T = \gamma'(t) = e_1$

At $\gamma(0)$, extend to an orthonormal basis for $T_{\gamma(0)}M$.

By parallel transport, $\nabla_{\frac{\partial\gamma}{\partial t}} e_j = 0 \rightsquigarrow$ orthonormal trivializing sections of $TM|_{\gamma(t)}$

3° $V = v^i(t)\, e_i$. $\nabla_{\frac{\partial\gamma}{\partial t}} V = \dot{v}^i(t)\, e_i$. $\nabla_{\frac{\partial\gamma}{\partial t}}\nabla_{\frac{\partial\gamma}{\partial t}} V = \ddot{v}^i(t)\, e_i$

The Jacobi equation reads

$$\ddot{v}^i(t)\, e_i = R(e_0, v^j(t) e_j)\, e_0 = R(e_0, e_j)\, e_0\big|_{\gamma(t)} \cdot v^j(t)$$

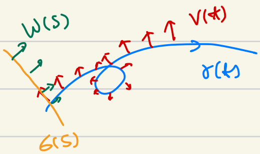
It is second order linear ODE.

$\Rightarrow$ solution exists and is determined by $V(0)$ and $\nabla_{\frac{\partial\gamma}{\partial t}} V\big|_0 = \dot{V}(0)$

4° Cor A Jacobi field always arises as a variational field of geodesics

pf: $V(t)$: Jacobi field along a geodesic $\gamma(t)$.

$V(0),\ \nabla_{\frac{\partial\gamma}{\partial t}} V\big|_{t=0} \left(= \tilde{\nabla}_{\frac{\partial}{\partial t}} V\right) \in T_{\gamma(0)}M$



Let $\sigma(s) = \exp_p(s\, V(0))$. curve through $\sigma(0)$ with $\sigma'(0) = V(0)$

Extend $V(0)$ smoothly along $\sigma(s)$: $W(s)$.

Require that $W(0) = V(0)$, $(\nabla_{\frac{\partial\sigma}{\partial s}} W)\big|_{s=0} = (\nabla_{\frac{\partial\gamma}{\partial t}} V)\big|_{t=0}$

Let $r(t,s) = \exp_{\sigma(s)}(t\, W(s))$. $\hat{V}(t) = \frac{\partial r}{\partial s}\Big|_{s=0}$

- $\gamma(t,s)$: geodesics $\forall s$ $\Rightarrow$ $\hat{V}(t)$: Jacobi field
- $\hat{V}(0) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_{\sigma(s)}(0) = \frac{\partial}{\partial s}\Big|_{s=0} \sigma(s) = V(0)$
- $\nabla_{\frac{\partial\gamma}{\partial t}} \hat{V}\Big|_{t=0} = \left(\nabla_{\frac{\partial\gamma}{\partial t}} \frac{\partial\gamma}{\partial s}\Big|_{s=0}\right)\Big|_{t=0} = \Big(\nabla_{\frac{\partial\sigma}{\partial s}} \underbrace{\frac{\partial\gamma}{\partial t}\Big|_{t=0}}_{=\ (\exp_{\sigma(s)})_*|_0 (W(s)) = W(s)}\Big)\Big|_{s=0} = \left(\nabla_{\frac{\partial\sigma}{\partial s}} W(s)\right)\Big|_{s=0} = \left(\nabla_{\frac{\partial\gamma}{\partial t}} V\right)\Big|_{t=0}$

Hence, $\hat{V}(t)$ and $V(t)$ satisfies the same ODE with the same initial condition. $\Rightarrow V(t) \equiv \hat{V}(t)$ #

5° remark • Jacobi equation $\Rightarrow$ when $K > 0$, geodesics will close up in the infinitesimal sense
($\exists\, t_0 > 0 \rightarrow V(t_0) = 0$)

- $\exp_p(x^i e_i)$, $g_{ij}(x) = \delta_{ij} +$ expansion at 0

$\boxed{\text{HW}}$ can be found by Jacobi equation

§IV. second variational formula (of energy)

$\gamma : \underset{t}{[0,1]} \times \underset{s}{(-\varepsilon, \varepsilon)} \longrightarrow M$ (assume $\gamma_s(t)$ is smooth)

$$\frac{d}{ds} E[\gamma_s] = -\int_0^1 \left\langle \frac{\partial\gamma}{\partial s}, \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial t} \right\rangle dt + \left\langle \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t} \right\rangle \Big|_{t=0}^{t=1}$$

(geodesic ~ where $E' = 0$ ⇝ second derivative test E'')

Assume $\gamma_0(t)$ is a geodesic ; calculate $\frac{d^2}{ds^2}\Big|_{s=0} E[\gamma_s]$

- $-\frac{d}{ds}\Big|_{s=0} \int_0^1 \left\langle \frac{\partial\gamma}{\partial s}, \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial t} \right\rangle dt$

$\gamma_0(t) = $ geodesic

$$= -\int_0^1 \left\langle \tilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial\gamma}{\partial s}, \underbrace{\cancel{\tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial t}}}_{0} \right\rangle dt - \int_0^1 \left\langle \frac{\partial\gamma}{\partial s}, \tilde{\nabla}_{\frac{\partial}{\partial s}} \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial t} \right\rangle dt$$

$$-\tilde{\nabla}_{\frac{\partial}{\partial s}} \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial t} = F_{\tilde{\nabla}}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) \frac{\partial\gamma}{\partial t} - \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial\gamma}{\partial t}$$

$$= R\left(\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s}\right) \frac{\partial\gamma}{\partial t} - \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial s}$$

$$= \int_0^1 \left\langle R\left(\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s}\right) \frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s} \right\rangle dt - \int_0^1 \underbrace{\left\langle \frac{\partial\gamma}{\partial s}, \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial s} \right\rangle}_{= \frac{\partial}{\partial t} \left\langle \frac{\partial\gamma}{\partial s}, \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial s} \right\rangle - \left\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial s}, \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial s} \right\rangle} dt$$

$$= \int_0^1 \left(\left\langle R\left(\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s}\right) \frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s} \right\rangle + \left| \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial s} \right|^2 \right) dt - \cancel{\left\langle \frac{\partial\gamma}{\partial s}, \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial s} \right\rangle} \Big|_{(0,0)}^{(1,0)}$$

- $\frac{d}{ds}\Big|_{s=0} \left(\left\langle \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t} \right\rangle \Big|_{t=0}^{t=1} \right) = \left\langle \tilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t} \right\rangle \Big|_{(0,0)}^{(1,0)} + \cancel{\left\langle \frac{\partial\gamma}{\partial s}, \tilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial\gamma}{\partial t} \right\rangle} \Big|_{(0,0)}^{(1,0)}$

Hence, $$\frac{d^2}{ds^2}\Big|_{s=0} E[\gamma_s] = \int_0^1 \left(\left| \nabla_{\frac{\partial\gamma}{\partial t}} \frac{\partial\gamma}{\partial s} \right|^2 - \left\langle R\left(\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s}\right) \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t} \right\rangle \right) dt + \left\langle \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t} \right\rangle \Big|_{(0,0)}^{(1,0)}$$

remark There are many ways to use it. Here is one example.

$\langle R(U,V)V, U\rangle > 0$ on spheres

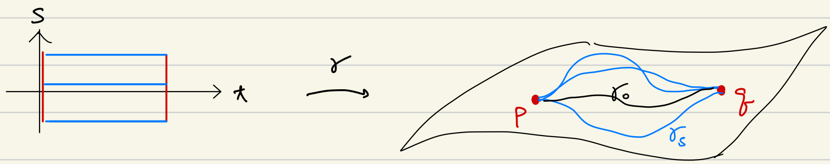
$\quad\text{"} \qquad < 0$ on hyperbolic spaces

(M,g) is said to have negative sectional curvature if

$\langle R(U,V)V, U\rangle < 0 \quad \forall p \in M,\ U, V \in T_pM$

(linearly independent)

If so, geodesics are locally energy / length minimizing



$\gamma_s(0) = p$

$\gamma_s(1) = q$

$\gamma_0(t)$: geodesic

$$\gamma_s(0) = p \Rightarrow \left.\frac{\partial \gamma}{\partial s}\right|_{(0,s)} \equiv 0 \Rightarrow \left.\tilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial s}\right|_{(0,0)} = 0$$

$\rightsquigarrow$ no boundary terms in the 2^{nd} variational formula for fixed endpoints family

$$\Rightarrow \left.\frac{d^2}{ds^2}\right|_{s=0} E[\gamma_s] = \int_0^1 \left|\tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s}\right|^2 \underbrace{- \left\langle R\left(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s}\right)\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t}\right\rangle}_{>0} dt > 0$$

※

(unless $\frac{\partial \gamma}{\partial s} \parallel \frac{\partial \gamma}{\partial t}$ along γ_0
but physically, it does not deform γ_0)