§I. length and energy

$0 = t_0 < t_1 < \cdots < t_N = 1$
$\gamma : (t_j, t_{j+1}) \to M$ is smooth

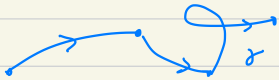
(M, g) : Riemannian manifold

1° $\gamma : [0,1] \to M$, continuous and piecewise smooth

The arc-length is defined to by

$$L[\gamma] = \int_0^1 |\gamma'(t)| \, dt$$

- It does not depend on the parametrization, only depend on its image in M.
- We always assume $\lim_{t \to t_j^{\pm}} \gamma'(t)$ exists, $\gamma'(t) \neq 0$



2° It is harder to do calculus of variation on the length, but is easier on the energy:

$$E[\gamma] = \frac{1}{2}\int_0^1 |\gamma'(t)|^2 \, dt$$

price we pay: $E[\gamma]$ depends on the parametrization.

This is not a big deal. By Cauchy-Schwarz,

$$\left(\int_0^T |\gamma'(t)| \, dt\right)^2 \le \left(\int_0^1 |\gamma'(t)|^2 dt\right)\left(\int_0^1 1 \, dt\right)$$

$$\Rightarrow \quad L[\gamma] \le \sqrt{2 \cdot E[\gamma]}$$

With "=" holds if and only if $|\gamma'(t)| = \text{constant}$

γ is parametrized by arc-length (up to some multiple)

3° prop $\gamma : \gamma(0) = p$, $\gamma(1) = q$, $|\gamma'(t)| = \text{constant}$

Then γ minimizes L among $\Omega_{p,q}$ if and only if it minimizes E among $\Omega_{p,q}$

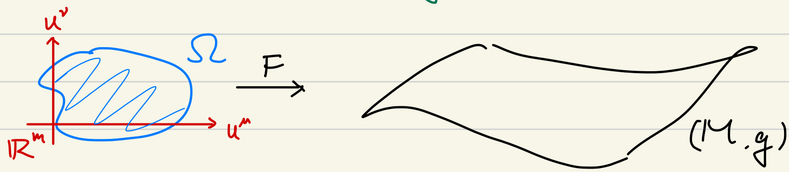
pf: $\Rightarrow)$ σ : another curve with $\sigma(0)=p$, $\sigma(1)=q$.

Known: $\underbrace{L[\gamma]}_{=\sqrt{2E[\gamma]}} \le \underbrace{L[\sigma]}_{\le\sqrt{2E[\sigma]}}$

$\Leftarrow)$ $\tilde\sigma$: anothe curve with $\tilde\sigma(0)=p$, $\tilde\sigma(1)=q$. σ: same curve with $|\sigma'|=$ constant

Known: $\underbrace{E[\gamma]}_{=\frac12(L[\gamma])^2} \le E[\sigma] = \frac12(L[\sigma])^2 = \frac12(L[\tilde\sigma])^2$ #

§II. critical state of energy

1° 

∇ : Levi-Civita

$\tilde\nabla = F^*\nabla$ is a metric connection on F^*TM (over Ω)

Locally, F means that $x=x(u)$

w.r.t. $\{\frac{\partial}{\partial x^i}\}$, $\nabla = d + \Gamma^j_{ik}\,dx^k$

$\rightsquigarrow \tilde\nabla = d + \Gamma^j_{ik}(x(u))\frac{\partial x^k}{\partial u^\mu}du^\mu$

<u>key properties</u> $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathfrak{X}(\Omega)$

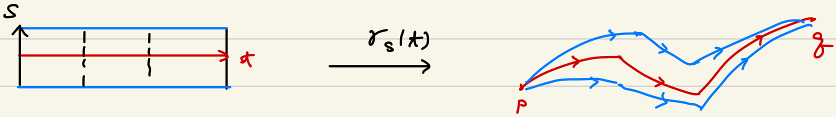
$\rightsquigarrow F_*(\mathcal{U}), F_*(\mathcal{V}), F_*(\mathcal{W}) \in \Gamma(F^*TM)$

(i) $\mathcal{U}\langle F_*(\mathcal{V}), F_*(\mathcal{W})\rangle = \langle\tilde\nabla_{\mathcal{U}}F_*(\mathcal{V}), F_*(\mathcal{W})\rangle + \langle F_*(\mathcal{V}), \tilde\nabla_{\mathcal{U}}F_*(\mathcal{W})\rangle$

(ii) $\tilde\nabla_{\mathcal{U}}F_*(\mathcal{V}) - \tilde\nabla_{\mathcal{V}}F_*(\mathcal{U}) = F_*([\mathcal{U},\mathcal{V}])$

$\boxed{\text{check}}$ direct calculation in coordinates.

We often abuse notation, and write $\nabla_{F_*(\mathcal{U})}F_*(\mathcal{V})$ for $\tilde\nabla_{\mathcal{U}}F_*(\mathcal{V})$

2° 

$\gamma_0(t)$: minimize E in $\Omega_{p,q}$ (assume $|\gamma_0'(t)|=$ constant)

$0 = \left.\frac{d}{ds}\right|_{s=0} E[\gamma_s(t)] \rightsquigarrow$ 1st variational formula for the energy functional

$$0 = \frac{d}{ds}\Big|_{s=0} E[\gamma_s(t)] = \frac{1}{2}\frac{d}{ds}\int_0^1 \left\langle \frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial t}\right\rangle dt$$

$$= \int_0^1 \left\langle \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial t}\right\rangle dt$$

$$= \int_0^1 \left\langle \nabla_{\frac{\partial\gamma}{\partial t}} \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right\rangle dt = \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} \left\langle \nabla_{\frac{\partial\gamma}{\partial t}} \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right\rangle dt$$

1st variational formula

$$= \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\left(\frac{\partial}{\partial t}\left\langle \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right\rangle - \left\langle \frac{\partial\gamma}{\partial s}, \nabla_{\frac{\partial\gamma}{\partial t}}\frac{\partial\gamma}{\partial t}\right\rangle\right) dt$$

$$= -\int_0^1 \left\langle V, \nabla_{\frac{\partial\gamma}{\partial t}}\frac{\partial\gamma}{\partial t}\right\rangle dt + \left\langle V(1), \frac{\partial\gamma}{\partial t}(1^-)\right\rangle - \left\langle V(0), \frac{\partial\gamma}{\partial t}(0^+)\right\rangle$$

Denote $\frac{\partial\gamma}{\partial s}\Big|_{s=0}$ by $V(t)$

$$+ \sum_{j=1}^{N-1}\left\langle V(t_j), \frac{\partial\gamma}{\partial t}(t_j^-) - \frac{\partial\gamma}{\partial t}(t_j^+)\right\rangle$$

The formula is true for any variation, namely, any V.

(i) Take $V = \rho(t)\nabla_{\frac{\partial\gamma}{\partial t}}\frac{\partial\gamma}{\partial t}$ $\quad \rho(t) \geq 0$. $\rho(t_j) = 0$ $j \in \{0, 1, \cdots, N\}$

$$\Rightarrow 0 = \int_0^1 \rho(t)\left|\nabla_{\frac{\partial\gamma}{\partial t}}\frac{\partial\gamma}{\partial t}\right|^2 dt$$

$$\Rightarrow \nabla_{\frac{\partial\gamma}{\partial t}}\frac{\partial\gamma}{\partial t} = 0 \text{ on each } (t_{j-1}, t_j)$$

(ii) Then, take V with $V(0) = 0 = V(1)$

$V(t_j) = \frac{\partial\gamma}{\partial t}(t_j^-) - \frac{\partial\gamma}{\partial t}(t_j^+)$ $\quad j \in \{1, \cdots, N-1\}$

$$\Rightarrow 0 = \sum_{j=1}^{N-1}\left|\frac{\partial\gamma}{\partial t}(t_j^-) - \frac{\partial\gamma}{\partial t}(t_j^+)\right|^2$$

$\Rightarrow \gamma$ is C^1

3° <u>geodesic equation</u>. recall that $\nabla_{\frac{\partial\gamma}{\partial t}}\frac{\partial\gamma}{\partial t}$ means $\tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial\gamma}{\partial t}$

$\gamma = (x^1(t), \cdots, x^n(t))$ $\qquad \frac{\partial\gamma}{\partial t} = \dot{x}^k\frac{\partial}{\partial x^k}$

$$\tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial\gamma}{\partial t} = \tilde{\nabla}_{\frac{\partial}{\partial t}}\left(\dot{x}^k\frac{\partial}{\partial x^k}\right) = \left(\ddot{x}^k + \Gamma^k_{ij}(x(t))\,\dot{x}^i\dot{x}^j\right)\frac{\partial}{\partial x^k}$$

<u>defn</u> a curve satisfies $\ddot{x}^k + \Gamma^k_{ij}(x(t))\, \dot{x}^i \dot{x}^j = 0$
is called a geodesic

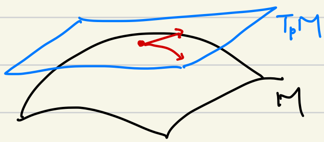
<u>lemma</u> ⓘ a geodesic must have constant velocity $\boxed{HW}$
ⓘⓘ $\forall p \in M,\ v \in T_pM$
There exists a unique geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$
with $\gamma(0) = p$, $\gamma'(0) = v$ (by ODE)

<u>lemma*</u> $\gamma(t) : (-\varepsilon, \varepsilon) \to M$ is a geodesic if and only if
$\gamma(ct) : (-\frac{\varepsilon}{c}, \frac{\varepsilon}{c}) \to M$ is a geodesic $(\forall c > 0)$

pf: chain rule on the geodesic equation ※

§ III. (geodesic) exponential map

1° Fix $p \in M$. choose an (orthonormal) basis $\{e_j\}$ for T_pM



For any $\{\underline{x}^j\} \in \mathbb{R}^n$ with $\sum_j (\underline{x}^j)^2 = 1$,
linear coordinate on T_pM given by $\{e_j\}$

Consider geodesic γ with
$\gamma(0) = p$, $\gamma'(0) = \underline{x}^j e_j$

By ODE, it is defined for $t \in (-\varepsilon, \varepsilon)$ where $\varepsilon > 0$ depends on $\{\underline{x}^j\}$
Since S^{n-1} is compact, we can find a uniform range $\varepsilon_p > 0$

2° Together with the above lemma*. we have

$$\exp_p : B(0; \varepsilon_p) \subset T_pM \longrightarrow M$$
$$v \longmapsto \exp_p(v)$$

$\gamma(1)$ where $\gamma(0) = p$, $\gamma'(0) = v$

By ODE (smooth dependence on the initial condition)
$\exp_p$ is a smooth map

3°. differential of the exponential map at the origin ?

$$(\exp_p)_*\big|_0 : T_0(T_pM) \cong T_pM \longrightarrow T_pM$$

$$v \longmapsto ⓘ$$

(Say, $|v| < \varepsilon_p$)

$v \rightsquigarrow \sigma(s) = sv \in T_pM$, $\sigma(0) = 0$, $\sigma'(0) = v$

$$ⓘ = \frac{d}{ds}\Big|_{s=0} \exp_p(\sigma(s)) = \frac{d}{ds}\Big|_{s=0} \exp_p(sv)$$

Let $\gamma(t)$ be the geodesic with $\gamma(0) = 0$, $\gamma'(0) = v \Rightarrow \exp_p(v) = \gamma(1)$

By lemma*, $\tilde{\gamma}_s(t) = \gamma(st)$ is a geodesic with $\tilde{\gamma}_s(0) = p$. $\tilde{\gamma}_s'(0) = sv$

$\Rightarrow \exp_p(sv) = \tilde{\gamma}_s(1) = \gamma(s)$ Hence, ⓘ $= v$

Namely, $(\exp_*)\big|_0$ is the identity map.

<u>prop</u> $\forall p \in M$, $\exists \varepsilon_p > 0$ such that

$\exp_p : B(0;\varepsilon_p) \subset T_pM \to M$ is an embedding

Thus it gives a local coordinate

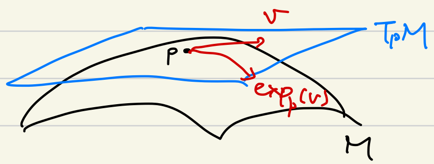
<u>rmk</u> $v \in B(0;\varepsilon_p)$, $\gamma(t) = \exp_p(tv)$ $t \in [0,1]$ is a geodesic

$|\gamma'(t)| = |v| \Rightarrow L[\gamma] = |v|$

§ IV. Gauss lemma

main goal : radial geodesic gives the distance

0°



$\forall p, q \in M$. define $d(p,q)$ to be $\inf\limits_{\gamma \in \Omega_{p,q}} L[\gamma]$

($\triangle$-inequality is easy to check)

<u>prop</u>☆ (shrink ε_p a little bit) $\forall v \in \overline{B(0;\varepsilon_p)}$

$d(p, \exp_p(v)) = |v|$

Thus, the distance is achived by radial geodesic (and unique) open $\mathcal{U} \subset M$

same notation as connection

1° <u>defn</u> $f \in C^\infty(\mathcal{U})$, $\nabla f \in \mathfrak{X}(M)$ is the metric dual of df

Namely, $\langle \nabla f, V \rangle = (df)(V) = V(f) \quad \forall\, V \in \mathfrak{X}(\mathcal{U})$

It is called the gradient of f

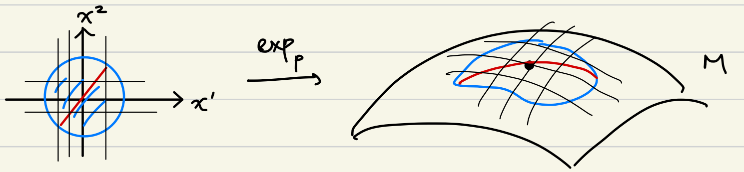
<u>e.g.</u> $(M, g) = (\mathbb{R}^n, \sum_{j=1}^n (dx^j)^2)$ $\quad df = \partial_j f\, dx^j \rightsquigarrow \nabla f = \partial_j f \frac{\partial}{\partial x^j}$

(as in calculus)

2° <u>lemma</u> (Gauss lemma) Use the geodesic normal coordinate

$\exp_p(x^j e_j)$. let $r = \sqrt{\sum_j (x^j)^2}$ (smooth except at p)

Then, $\nabla r = \frac{1}{r} x^j \frac{\partial}{\partial x^j} \quad \left(= \frac{\partial}{\partial r}\right)$



<u>discussion</u> $x^j \frac{\partial}{\partial x^j} \in T_x \mathbb{R}^n$ (radial vector field on $\mathbb{R}^n$)

$\longleftrightarrow (\exp_p)_* \big|_x \left(x^j \frac{\partial}{\partial x^j}\right)$ (real meaning of the notation)

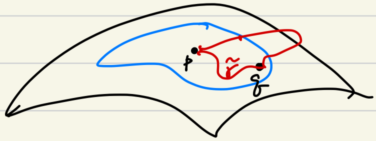
For any (fixed) $(x^1, \cdots, x^n)$, $\sigma(t) = (tx^1, \cdots, tx^n)$

$$(\exp_p)_* \Big|_x \left(x^j \frac{\partial}{\partial x^j}\right) = \frac{d}{dt}\Big|_{t=1} \exp_p(\sigma(t)) = \frac{d}{dt}\Big|_{t=1} \underbrace{\exp(tx)}_{=\gamma(t)}$$

Since $\exp_p(tx)$ is a geodesic, $|\gamma'(1)|^2 = |\gamma'(0)|^2 = \sum (x^j)^2$

Hence, $\frac{1}{r} x^j \frac{\partial}{\partial x^j}$ has constant length 1

3° pf of the <u>prop</u>*:



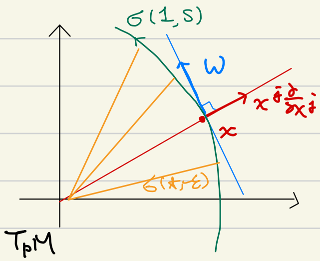
$\tilde{\gamma} \in \Omega_{p,q} \quad \tilde{\gamma} \subset \exp_p(B(0;\varepsilon))$

$$\begin{aligned} L[\tilde{\gamma}] &= \int_0^1 |\tilde{\gamma}'|\, dt = \int_0^1 |\tilde{\gamma}'|\,|\nabla r|\, dt \\ &\geq \int_0^1 \langle \nabla r, \tilde{\gamma}' \rangle\, dt \\ &= \int_0^1 dr(\tilde{\gamma}')\, dt = \int_0^1 \frac{d}{dt}(r \circ \tilde{\gamma})\, dt \\ &= r(q) \quad \#\end{aligned}$$

$\frac{1}{r} x^j \frac{\partial}{\partial x^j}$

$\Downarrow$

4° proof of the Gauss lemma $\langle \nabla r, W \rangle = dr(W)$ $\quad dr = \frac{1}{r} \sum_j x^j dx^j$



(x : fixed and non zero)

$W = \sum_j W^j \frac{\partial}{\partial x^j} \perp$ position (in T_pM) $\Leftrightarrow \sum_j W^j x^j = 0$

If we can show $(\exp_p)_*|_x (W) \perp (\exp_p)_*|_x (x^j \frac{\partial}{\partial x^j})$ when $\sum_j W^j x^j = 0$

then $\nabla r \parallel (\exp_p)_*|_x (x^j \frac{\partial}{\partial x^j})$

(The argument in 2° can pin down the length)

Consider $\sigma(t, s) = t \cdot \left(\cos s \, \vec{x} + \sin s \, \vec{w} \frac{|\vec{x}|}{|\vec{w}|} \right) : [0, 1] \times (-\varepsilon, \varepsilon) \to T_pM$

let $\gamma_s(t) = \exp_p(\sigma(t, s))$.

For each s, $\gamma_s(t)$ is a geodesic, $\quad E[\gamma_s(t)] = \frac{1}{2} |\vec{x}|^2$

$L[\gamma_s(t)] = |\vec{x}|$

$\Rightarrow 0 = \left. \frac{d}{ds} \right|_{s=0} E[\gamma_s(t)]$

by 1st variational formula

$= \langle V(1), \gamma'(1) \rangle$

$V(1) = (\exp_p)_*|_x \left(\vec{w} \frac{|\vec{x}|}{|\vec{w}|} \right)$, $\quad \gamma'(1) = (\exp_p)_*|_x (\vec{x})$ ※

(general trick: interpret vector field as variational field of geodesics, see what the variational formula tells us)