§0. introduction

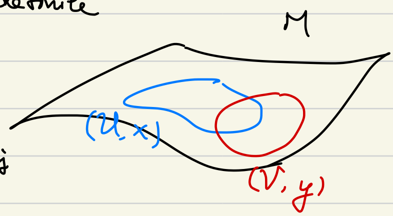
Riemannian geometry : study the geometry related to distance and angle

M : manifold (usually without boundary)

$\rightsquigarrow TM$: canonically defined on M.

Endow TM a bundle metric

Namely, $\sum_{ij} g_{ij}(x)\, dx^i \otimes dx^j$ $\quad [g_{ij}]$: positive definite



$$g_{ij}\, dx^i \otimes dx^j \overset{?}{=} \tilde{g}_{kl}(y)\, dy^k \otimes dy^l$$

$$= \tilde{g}_{kl}(y(x)) \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j}\, dx^i \otimes dx^j$$

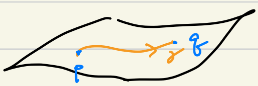
$$g_{ij}(x) = \sum_{kl} \tilde{g}_{kl}(y(x)) \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j}$$

$\Gamma(\mathrm{Sym}(T^*M \otimes T^*M))$

$\quad\quad \Downarrow$

<u>defn</u> This is called a Riemannian manifold. (M, g)

- curve

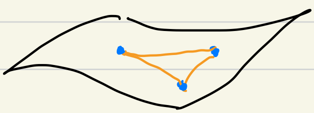


Length $= \int |\gamma'(t)|\, dt$

$\hookrightarrow \gamma(t) = \{x^j(t)\}$ $\quad |\gamma'(t)|^2 = g_{ij}(x(t)) \frac{dx^i}{dt} \frac{dx^j}{dt}$

curve of shortest distance ?

-



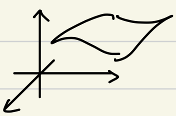
cosine formula ?

correction by curvature ?

$\hookrightarrow$ of a metric connection on TM

$\hookrightarrow$ which connection ?

- example



$M \subset \mathbb{R}^N \rightsquigarrow T_pM \subset \mathbb{R}^N$

restrict the standard metric

more generally $\quad M \subset (\tilde{M}, \tilde{g}) \rightsquigarrow T_pM \subset (T_p\tilde{M}, \tilde{g})$

restrict the metric $\quad (M, g)$

relation between ambient and subspace geometry ?

§I. Levi-Civita connection

1° ∇ : any connection on TM

special point of TM : a vector field can be the direction of taking derivative, and also be differentiated

$$\rightsquigarrow \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$
$$(X, Y) \longmapsto \nabla_X Y - \nabla_Y X$$

tensorial ? $(f_1 X, f_2 Y) \mapsto \nabla_{f_1 X} f_2 Y - \nabla_{f_2 Y} f_1 X$

$$= f_1 f_2 (\nabla_X Y - \nabla_Y X) + f_1 \cdot X(f_2) \cdot Y - f_2 \cdot Y(f_1) \cdot X$$

recall : $(X, Y) \longmapsto [X, Y]$

$(f_1 X, f_2 Y) \mapsto (f_1 X)(f_2 Y) - (f_2 Y)(f_1 X)$

$$= f_1 f_2 [X, Y] + f_1 \cdot X(f_2) \cdot Y - f_2 \cdot Y(f_1) \cdot X$$

<u>lem</u> ∇ : a connection on TM. Then,

$(X, Y) \longmapsto \nabla_X Y - \nabla_Y X - [X, Y]$ is a tensor,

which is called the torsion of ∇ $\Gamma(\mathrm{Hom}(TM \otimes TM ; TM))$

2° <u>thm</u> (basic theorem in Riemannian geometry)

(M, g) : Riemannian. Then, there exists a unique ∇ on TM which is a torsion free, metric connection

pf: $X, Y, Z \in \mathfrak{X}(M)$

$$+\ X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$
$$+\ Y(g(Z, X)) = g(\nabla_Y Z, X) + g(Z, \underbrace{\nabla_Y X}_{= \nabla_X Y - [X, Y]})$$
$$-\ Z(g(X, Y)) = g(\underbrace{\nabla_Z X}_{= \nabla_X Z + [Z, X]}, Y) + g(X, \underbrace{\nabla_Z Y}_{= \nabla_Y Z + [Z, Y]})$$

$$X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y))$$
$$= 2g(\nabla_X Y, Z) - g([X, Y], Z) - g([Z, X], Y) + g([Y, Z], X)$$

explain more in class

$$\Rightarrow g(\nabla_X Y, Z) = \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X) \Big)$$

$\boxed{\text{check}}$ right hand side is function linear in X and Z ※

3° <u>main</u> <u>example</u> $M \subset \mathbb{R}^N \rightsquigarrow T_pM \subset \mathbb{R}^N$

restrict the standard inner product

vector field $\subset \mathcal{C}^\infty(M;\mathbb{R}^N)$

<u>claim</u> the Levi-Civita connection is exactly $\mathrm{proj}\circ d_{\mathbb{R}^N}$

ⓘ metric connection $X, Y : M \to \mathbb{R}^N$

$X(p), Y(p) \in T_pM$

$$d\langle X,Y\rangle = \langle dX, Y\rangle + \langle X, dY\rangle$$
$$= \langle \mathrm{proj}\circ dX, Y\rangle + \langle X, \mathrm{proj}\circ dY\rangle$$

ⓘⓘ torsion free

<u>basic</u> <u>property</u> (from chain rule) $X \in \mathfrak{X}(M)$, $f \in \mathcal{C}^\infty(M)$

fix $p \in M$. choose a small ball in $\mathbb{R}^N$ at p

and any smooth extension of X, $\tilde{X} : \mathcal{U} \to \mathbb{R}^N$

of f, $\tilde{f} : \mathcal{U} \to \mathbb{R}^1$

Then, $\underbrace{X(f)}_{\text{on } M}\big|_p = X(\tilde{f})\big|_p = \tilde{X}(\tilde{f})\big|_p$

Now, $X, Y \in \mathfrak{X}(M)$, $f \in \mathcal{C}^\infty(M)$

$$\Rightarrow [X,Y](f)\big|_p = [\tilde{X},\tilde{Y}](\tilde{f})\big|_p \quad \forall p \in M$$

$$\Rightarrow [\tilde{X},\tilde{Y}]\big|_p \in T_pM$$

(otherwise we can choose $\tilde{f}$: has non-trivial derivative in some normal)

But in $\mathbb{R}^3$. $[\tilde{X},\tilde{Y}] = \iota(\tilde{X})\, d(\tilde{Y}) - \iota(\tilde{Y})\, d(\tilde{X})$

at $p \in M$ $\quad \mathrm{proj}([\tilde{X},\tilde{Y}]|_p) = [\tilde{X},\tilde{Y}]\big|_p$

$$= \mathrm{proj}\big((\iota(X)\, d(\tilde{Y}) - \iota(Y)\, d\tilde{X})\big)\big|_p$$

$$= (\nabla_X Y - \nabla_Y X)\big|_p$$

use
- basic property
- from definition of $[\ ,\ ]$ for vector fields on $\mathbb{R}^N$

4° a coordinate form of 2°

$(\mathcal{U}, \{x^i\})$ $\quad g = g_{ij}(x)\, dx^i \otimes dx^j$ $\quad$ intrinsic coordinate from 1 to n

$\nabla : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM)$

$\nabla \frac{\partial}{\partial x^j} = \sum_{i,k} \Gamma^k_{ij}\, dx^i \otimes \frac{\partial}{\partial x^k}$ $\quad$ namely, $\quad \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}$

Take $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$ $Z = \frac{\partial}{\partial x^l}$

$\sum_k g_{kl} \Gamma^k_{ij} = \frac{1}{2}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ $\quad g^{ji} = g^{ij} = (g^{-1})_{ij}$

$\sum_l g^{l\tilde{k}}$ $\quad \sum_j g^{ij} g_{jk} = \delta^i_k$

$$\Gamma^{\cancel{k}}_{ij} = \frac{1}{2} \sum_l g^{\cancel{k}l} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

note that $\Gamma^k_{ij} = \Gamma^k_{ji}$

§II. curvature tensor

0° recall $\quad \nabla = d + A$ on E, $A \in \Omega^1(\mathrm{End}\, E)$

from the consideration of commuting derivatives

$F_\nabla = dA + A \wedge A \in \Omega^2(\mathrm{End}\, E) = \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}\, E)$

recall $\quad E$: real vector bundle with metric $\Rightarrow E \cong E^*$

Now, $E = TM$, $E^* = T^*M$

$\{\frac{\partial}{\partial x^i}\}$ $\quad \{dx^i\}$

$dx^i \longmapsto a^{ij} \frac{\partial}{\partial x^j}$ defined by $dx^i(\frac{\partial}{\partial x^k}) = \langle \frac{\partial}{\partial x^k}, a^{ij} \frac{\partial}{\partial x^j} \rangle$

$\delta^i_k = g_{kj} a^{ij}$

Hence, $a^{ij} = g^{ij}$

Also, $\frac{\partial}{\partial x^i} \longmapsto g_{ij}\, dx^j$

Now, $\quad \mathrm{End}\, TM = TM \otimes T^*M \cong T^*M \otimes T^*M$

1° $\nabla = d + \Gamma^k_{ij} dx^j$ $\quad$ trivialize TM by $\{\frac{\partial}{\partial x^i}\}$

Endomorphism

1-form

$$F = dA + A\wedge A = \frac{1}{2} F_{ij}\, dx^i \wedge dx^j$$

$$\text{where} \quad F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} + [A_i, A_j]$$

$$A_i = \Gamma^l_{ki}$$

$$(F_{ij})\left(\frac{\partial}{\partial x^k}\right) = \sum_l R_k{}^l{}_{ij} \frac{\partial}{\partial x^l} = \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} - \nabla_{\cancelto{0}{[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]}} \frac{\partial}{\partial x^k}$$

$$R^l_{kij} = \frac{\partial}{\partial x^i} \Gamma^l_{kj} - \frac{\partial}{\partial x^j} \Gamma^l_{ki} + \Gamma^m_{ki}\Gamma^l_{mj} - \Gamma^m_{kj}\Gamma^l_{mi}$$

2° $R^l_{kij}(dx^i \wedge dx^j) \otimes (dx^k \otimes \frac{\partial}{\partial x^l}) \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(TM))$

is called the Riemann curvature tensor

$$R_{lkij} = \sum_m g_{lm} R^m_{kij} \rightsquigarrow R_{lkij}(dx^i \wedge dx^j) \otimes (dx^k \otimes dx^l)$$

$\in$ $\Gamma(\Lambda^2 T^*M \otimes \Lambda^2 T^*M)$

<u>lemma-C1</u> $R_{lkij} = -R_{klij}$

pf: Fix i, j (and omit them): this is linear algebra

$F(\frac{\partial}{\partial x^k}) = R^m_k \frac{\partial}{\partial x^m}$ Since ∇ is a metric connection F is a skew-adjoint operator

$$\Rightarrow \left\langle F\left(\frac{\partial}{\partial x^k}\right), \frac{\partial}{\partial x^l}\right\rangle = -\left\langle \frac{\partial}{\partial x^k}, F\left(\frac{\partial}{\partial x^l}\right)\right\rangle$$

$$R_{lk} = g_{ml} R^m_k = -g_{mk} R^m_l = -R_{kl} \quad \#$$

3° More properties of curvature tensor

$$R(X,Y)U = \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U$$

Note that $R_{lkij} = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right\rangle$

<u>lemma-C2</u> $\langle R(X,Y)U, V\rangle = \langle R(U,V)X, Y\rangle$

Namely, $R_{lkij} = R_{jikl} = R_{ijlk}$ $\qquad \mathrm{Sym}(\Lambda^2 T^*M \otimes \Lambda^2 T^*M)$

<u>lemma-C3</u> (1st Bianchi identity of the Riemann curvature tensor)

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$$

pf of C3: $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X$

$$= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$+ \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y$$
$$+ \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X$$
$$= \nabla_X [Y,Z] - \nabla_{[Y,Z]} X + \nabla_Y [Z,X] - \nabla_{[Z,X]} Y$$
$$+ \nabla_Z [X,Y] - \nabla_{[X,Y]} Z$$
$$= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad ※$$

pf of C2: $\langle R(X,Y)U, V\rangle$

$$= -\langle R(U,X)Y, V\rangle - \langle R(Y,U)X, V\rangle \quad \text{by C3}$$
$$= \langle R(U,X)V, Y\rangle + \langle R(Y,U)V, X\rangle \quad \text{by C1}$$
$$= -\langle R(V,U)X, Y\rangle - \langle R(X,V)U, Y\rangle \quad \text{by C3}$$
$$- \langle R(V,Y)U, X\rangle - \langle R(U,V)Y, X\rangle$$
$$= 2\langle R(U,V)X, Y\rangle + \langle R(X,V)Y, U\rangle + \langle R(V,Y)X, U\rangle$$
$$= 2\langle R(U,V)X, Y\rangle - \langle R(Y,X)V, U\rangle \quad \text{by C3}$$
$$\Rightarrow \langle R(X,Y)U, V\rangle = \langle R(U,V)X, Y\rangle \quad ※$$

§III. Induced connection

1° ∇ on E $\rightsquigarrow$ induce ∇ on E^*

$\alpha \in \Gamma(E^*)$, $s \in \Gamma(E)$ require

$$(\nabla\alpha)(s) + \alpha(\nabla s) = d\,\alpha(s) \quad \in C^\infty(M)$$

$\alpha(s) = \alpha^T s$ as $\mathbb{R}^k$-valued functions

In terms of local trivialization. ∇ on $E = d + A$

$\rightsquigarrow \nabla$ on $E^* = d - A^T$

2° ∇ on E, ∇ on F $\rightsquigarrow$ ∇ on $\mathrm{Hom}(E,F)$

$T \in \Gamma(\mathrm{Hom}(E,F))$, $s \in \Gamma(E)$ $\Rightarrow$ $Ts \in \Gamma(F)$

require $\nabla(Ts) = (\nabla T)(s) + T(\nabla s)$

$$(\nabla T)(s) = \underset{F}{\nabla(Ts)} - T(\underset{E}{\nabla s})$$

In terms of local trivialization

$E|_U \cong U \times \mathbb{R}^k$, $F|_U \cong U \times \mathbb{R}^\ell$ $\Rightarrow$ $\mathrm{Hom}(E,F)|_U \cong M(\ell \times k; \mathbb{R})$

$\nabla = d + A_E$ $\quad \nabla = d + A_F$ $\Rightarrow$ $\nabla T = dT + A_F T - T A_E$

3° ∇ on E $\rightsquigarrow$ ∇ on $\otimes^k E$

$$\nabla_X(s_1 \otimes \cdots \otimes s_k) = (\nabla_X s_1) \otimes \cdots \otimes s_k + \cdots + s_1 \otimes \cdots \otimes \nabla_X s_k$$

also ∇ on $\Lambda^k E$

$$\nabla_X(s_1 \wedge \cdots \wedge s_k) = (\nabla_X s_1) \wedge s_2 \wedge \cdots \wedge s_k + \cdots + s_1 \wedge \cdots \wedge (\nabla_X s_k)$$

4° Now, focus on the Levi-Civita connection, and TM, T^*M (also from their algebraic constructions)

Fix $U \in \mathfrak{X}(M)$, $\nabla U \in \Gamma(\mathrm{End}(TM))$

$\rightsquigarrow \nabla^2 U = \nabla(\nabla U) \in \Gamma(T^*M \otimes \mathrm{End}(TM))$? explain more in class

Denote $(\nabla^2 U)(X,Y) = (\nabla_X(\nabla U))(Y)$ by $\nabla^2_{X,Y} U$

$= \nabla_X \nabla_Y U - (\nabla U)(\nabla_X Y)$ by 2°

$= \nabla_X \nabla_Y U - \nabla_{\nabla_X Y} U$

Hence, $\nabla^2_{X,Y} U - \nabla^2_{Y,X} U = R(X,Y)U$

rmk $\nabla_- \nabla_- U$ is not a tensor, but $(\nabla^2 U)(-,-)$ is.

5° lemma-C4 (2nd Bianchi identity of the Riemann curvature tensor)

$$(\nabla_X R)((Y,Z),U) + (\nabla_Y R)((Z,X),U) + (\nabla_Z R)((X,Y),U) = 0$$

rmk This is equivalent to $dF = F \wedge A - A \wedge F$ after unwinding all the definitions.