§I. Euler class

1° recall $\Sigma \subset \mathbb{R}^3$ with ∇ on $T\Sigma = \text{proj} \circ d_{\mathbb{R}^3}$

$$F_\nabla = \begin{bmatrix} 0 & KdA \\ -KdA & 0 \end{bmatrix}$$ w.r.t. $\{e_1, e_2\}$: oriented, orthonormal (local) tangent vector fields

tr $= 0$. $\det \in \Omega^4(\Sigma) = 0$

The theory of Chern class or Pontryagin class does not apply here. What is KdA?

note swith $e_1 \leftrightarrow e_2$ $F = \begin{bmatrix} 0 & -KdA \\ KdA & 0 \end{bmatrix}$

2° If we take orientation into consideration, there is one more invariant polynomial.

defn an $\mathbb{R}^k$-vector bundle is said to be oriented if $g_{uv} \in Gl^+(k;\mathbb{R})$ (the part with $\det > 0$)

rmk • Equivalently, $\Lambda^k E$ (also $\Lambda^k E^*$) is a trivial $\mathbb{R}^1$-bundle

• For $E = TM$, this is the orientation of manifold discussed last semester.

3° linear algebra consider even-rank, skew-symmetric matrices

$k=2$ $\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$ $k=4$ $\begin{bmatrix} 0 & a & d & f \\ -a & 0 & b & e \\ -d & -b & 0 & c \\ -f & -e & -c & 0 \end{bmatrix}$

$\det = a^2$ $\det = (ac - de + bf)^2$

defn $S \in M(2m \times 2m : \mathbb{R})$, $S^T = -S$

The Pfaffian of S is $$Pf(S) = \frac{1}{2^m m!} \sum_{S_{2m}} \text{sgn}(\sigma) S_{\sigma_1\sigma_2} \cdots S_{\sigma_{2m-1}\sigma_{2m}}$$

lem $\forall O \in O(2m)$, $Pf(OSO^{-1}) = \det O \cdot Pf(S)$

$(OSO^T)_{\sigma_1\sigma_2} = O_{\sigma_1 \tau_1} S_{\tau_1\tau_2} O_{\sigma_2\tau_2}$

$$\operatorname{sgn}(\sigma)(OSO^T)_{\sigma_1\sigma_2}\cdots(OSO^T)_{\sigma_{2m-1}\sigma_{2m}}$$

$$= \operatorname{sgn}(\sigma)(O_{\sigma_1 z_1} S_{z_1 z_2} O_{\sigma_2 z_2})(O_{\sigma_3 z_3} S_{z_3 z_4} O_{\sigma_4 z_4})\cdots(O_{\sigma_{2m-1} z_{2m-1}} S_{z_{2m-1} z_{2m}} O_{z_{2m}\sigma_{2m}})$$

$$= \operatorname{sgn}(z) S_{z_1 z_2}\cdots S_{z_{2m-1} z_{2m}} \operatorname{sgn}(\gamma = \sigma^{-1} z) O_{1\gamma_1}\cdots O_{2m\gamma_{2m}}$$

(some cancellations) ※

rmk For odd ranks, det = 0

4° defn For a rank $2m$, oriented vector bundle E, its Euler class is $e(E) = \left[\frac{1}{(2\pi)^m} \mathrm{Pf}(F_\nabla)\right] \in H^{2m}(M)$ for any choice of metric and metric connection.

§ II. topology of Euler class

1° prop Suppose that $E \to M$ is an oriented, rank $2m$ vector bundle. If E admits a nowhere zero section. Then $e(E) = 0$

pf: Call the section s, Endow E a bundle metric

$\Rightarrow E = \mathbb{R}\langle s\rangle \oplus \{s\}^\perp$

Since $\mathbb{R}\langle s\rangle$ is a trivial bundle, $\left(\frac{s}{|s|} \leftrightarrow \text{"}1\text{"}\right)$ the usual d gives a flat metric connection

Then, choose any metric connection ∇' on $\{s\}^\perp$.

They together define a metric connection on E

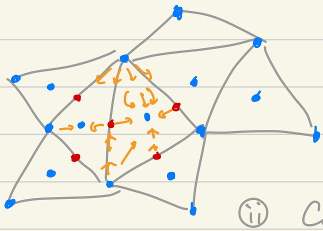
and $F = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & F_{\nabla'} & \\ 0 & & \end{bmatrix}$. Clearly, $\mathrm{Pf}(F) = 0$ ※

2° Euler class and Euler characteristic

For an oriented, closed manifold, call $e(TM)$ the Euler class of M.

prop $\displaystyle\int_M e(M) = \chi(M) = \sum_{j=1}^{2m} (-1)^j \dim H^j_{dR}(M)$

rmk When $\dim M = 2m+1$, $\chi(M) = 0$ by Poincaré duality $H^j_{dR}(M) \cong (H^{2m+1-j}_{dR}(M))^*$



idea of proof

ⓘ Choose a "triangulation" of M

$\chi(M) = E - V + F \pm \cdots$

by algebraic topology

ⓘⓘ Construct W: vector field on M

such that $\{\text{zeros of } W\} \leftrightarrow \{\text{center of edges/faces}\}$

Moreover, $\operatorname{ind}(W, p) = (-1)^{(\dim \text{ of the face})}$

ⓘⓘⓘ By modifying 1°, we can construct ∇ on TM

such that $e(M) = \dfrac{1}{(2\pi)^m} \operatorname{Pf}(F_\nabla)$ is supported

on small neighborhoods of W

ⓘⓥ local calculation $\displaystyle\int_{\text{nbd of } p} e(M) = \pm 1$ ※

§ III more topology of characteristic class

In some situation, the characteristic class can determine the topology (in a suitable sense).

$\dim M = n \equiv 0\ (4)$

M: oriented

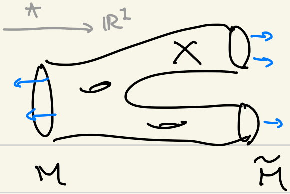
1° Pontryagin class $p_j(M) = p_j(TM) \in H^{4j}_{dR}(M)$

Pontryagin numbers $= \left\{ \displaystyle\int_M p_{j_1}(M) \wedge \cdots \wedge p_{j_\mu}(M) \;\middle|\; j_1 + \cdots + j_\mu = n \right\}$

2° M^n is said to be oriented cobordism to $\tilde{M}^n$ if

$\exists$ $(n+1)$-dim'l manifold with boundary X such that

$\partial X = (-M) \cup \tilde{M}$ (considering orientation)



induced orientation = contraction with outer normal

prop oriented cobordism $\Rightarrow$ all the Pontrygin numbers coincide

Choose ∇ on TX. $P_j(TX) = \sigma_{2j}(\frac{1}{2\pi}F_\nabla)$

$TX|_M = TM \oplus \underset{dt}{\mathbb{R}} \rightsquigarrow P_j(TM) = \sigma_{2j}(\frac{1}{2\pi}F_\nabla)\Big|_M$

$$0 = \int_X d\,P_J(TX) = \int_{\partial X} P_J(TX)\Big|_{\partial X} = -P_J(TM) + P_J(T\tilde{M}) \quad \#$$

thm (Thom, etc) Together with some $\mathbb{Z}/2$-data (Stiefel-Whitney numbers), the converse of the proposition holds true

This is deep. Thom was awarded the Fields medal for this work

§III. Some geometry aspects of the theory

Yang-Mills theory

many connections $\rightsquigarrow$ invariant polynomial (curvature) is topological quantity

"best" connection?

Say, there is an inner product on the curvatures

Try to minimize $\int_M |F_\nabla|^2 \, d\mathrm{vol}$ among all connections

e.g. $F = \sum_{i<j} F_{ij}\, dx^i \wedge dx^j \rightsquigarrow |F|^2 = \sum_{i<j} \langle F_{ij}, F_{ij} \rangle$

$$= \frac{1}{2} \sum_{i,j} \langle F_{ij}, F_{ij} \rangle$$

Suppose that ∇ minimizes $\int_M |F_\nabla|^2 \, d\mathrm{vol}$

$\forall B \in \Gamma(\Omega^1(M) \otimes \mathrm{End}(E))$

Consider $\nabla + tB \rightsquigarrow \frac{d}{dt}\Big|_{t=0} \int_M |F^t|^2 \, d\mathrm{vol} = 0$

$$t=0$$

Write $\nabla = d + A_i\, dx^i \rightsquigarrow F^\circ_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} + [A_i, A_j]$

$$\nabla + tB = d + (A_i + tB_i)\, dx^i$$

$$F^t_{ij} = F^\circ_{ij} + t\left(\frac{\partial B_j}{\partial x^i} - \frac{\partial B_i}{\partial x^j} + [B_i, A_j] + [A_i, B_j]\right) + t^2(\cdots)$$

$$\sum_{i,j} |F^t_{ij}|^2 = \sum_{i,j} |F^\circ_{ij}|^2 + 2t \left\langle F^\circ_{ij}, \frac{\partial B_j}{\partial x^i} - \frac{\partial B_i}{\partial x^j} + [B_i, A_j] + [A_i, B_j]\right\rangle + t^2(\cdots)$$

$$\frac{d}{dt}\Big|_{t=0} \int_M |F^t|^2\, d\mathrm{vol}$$

assume $d\mathrm{vol} = dx^1 \cdots dx^n$

$$\Rightarrow \int \sum_{i,j} \left\langle F^\circ_{ij}, \frac{\partial B_j}{\partial x^i} - \frac{\partial B_i}{\partial x^j} + [B_i, A_j] + [A_i, B_j] \right\rangle dx^1 \wedge \cdots \wedge dx^n$$

assume metric connection $\rightsquigarrow$ skew-symmetric matrices

$$\langle U, V \rangle = \operatorname{tr}(UV^*)$$

$$\langle U, \mathrm{ad}_W V \rangle = \operatorname{tr}(UV^*W^* - UW^*V^*) = -\langle \mathrm{ad}_W U, V \rangle$$

$$= \int \sum_{i,j} \left\langle -\frac{\partial F^\circ_{ij}}{\partial x^i}, B_j \right\rangle + \left\langle \frac{\partial F^\circ_{ji}}{\partial x^i}, B_j \right\rangle \quad \left(F^\circ_{ji} = -F^\circ_{ij}\right)$$

$$+ \langle [A_i, F^\circ_{ji}], B_j \rangle - \langle [A_i, F^\circ_{ij}], B_j \rangle \quad \left(F^\circ_{ji} = -F^\circ_{ij}\right)$$

$$= -2\int \sum_j \left\langle \sum_i \left(\frac{\partial F^\circ_{ij}}{\partial x^i} + [A_i, F^\circ_{ij}]\right), B_j \right\rangle = 0$$

True for any $B_j \Rightarrow \sum_i \left(\frac{\partial F^\circ_{ij}}{\partial x^i} + [A_i, F^\circ_{ij}] \right) = 0$

F° : curvature of A

the Yang–Mills equation

analogy: many curves $\rightsquigarrow$ critical states of length = geodesic

$\rightsquigarrow$ better theory / way to understand the space of "all curves"