§I. extract information from curvature

$F \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(E))$ — $k\times k$ matrices of 2-forms

For different trivializations, $g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta} = F_\beta$

For instance, $\mathrm{tr}(F_\alpha) = \mathrm{tr}(F_\beta)$

Hence, $\mathrm{tr}(F) \in \Omega^2(TM)$ is well-defined

It depends on ∇, but not the trivialization

1° a polynomial $f: M(k\times k; \mathbb{R}) \cong \mathbb{R}^{k^2} \to \mathbb{R}$ is called an <u>invariant polynomial</u> if $f(g^{-1}Bg) = f(B)$

$\forall\, B \in M(k\times k; \mathbb{R}),\ g \in GL(k;\mathbb{R})$

<u>lemma</u> Write $\det(\mathbb{I} + sB) = 1 + s\,\sigma_1(B) + \cdots + s^k \sigma_k(B)$

each $\sigma_j(B)$ is an invariant polynomial of deg j

Note that $\sigma_1(B) = \mathrm{tr}(B)$, $\sigma_k(B) = \det(B)$

pf: linear algebra ※

<u>fact</u> These are all invariant polynomials. (but we will not need this fact)

2° dF ? $\quad \nabla = d + A$

$$F = dA + A\wedge A$$

$$\Rightarrow dF = \cancel{d^2A}^{\,0} + dA\wedge A - A\wedge dA$$

$$= (F - A\wedge A)\wedge A - A\wedge(F - A\wedge A)$$

$$\Rightarrow dF = F\wedge A - A\wedge F$$

this is called the Bianchi identity

3° $\det(\mathbb{I} + sF) = 1 + s\,\sigma_1(F) + \cdots + s^k \sigma_k(F)$

$(\mathbb{I} + sF)^{-1} = \mathbb{I} - sF + s^2F^2 \mp \cdots + (-1)^j s^j F^j + \cdots$

vanishes after $j > \dim M/2$

recall:

$d(\det B) = (\det B)\ \mathrm{tr}(B^{-1} dB)$ by Cramer's rule

$\Rightarrow d(\det(\mathbb{I}+sF))$

$= (\det(\mathbb{I}+sF))\ \mathrm{tr}\left((\mathbb{I}+sF)^{-1} \wedge d(\mathbb{I}+sF)\right)$

$= (\det(\mathbb{I}+sF))\ \underbrace{\mathrm{tr}\left((\mathbb{I}-sF+s^2F^2 \mp \cdots + (-1)^j s^j F^j) \wedge s(F\wedge A - A\wedge F)\right)}_{\sum_j (-1)^{j+1} s^{j+1}\ \underbrace{\mathrm{tr}(F^j \wedge (F\wedge A - A \wedge F))}_{\mathrm{tr}(F^{j+1}\wedge A) - \mathrm{tr}(F^j \wedge A \wedge F)}}$

F: 2-forms
A: 1-form

$= \mathrm{tr}(F^{j+1}\wedge A) - \mathrm{tr}(F \wedge F^j \wedge A) = 0$

Therefore, $\sigma_j(F) = 0 \quad \forall j$.

4° sketch of another proof

Suppose that $B = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \Rightarrow \sigma_j(B) = \sum_{\alpha_1 < \cdots < \alpha_j} \lambda_{\alpha_1}\lambda_{\alpha_2}\cdots\lambda_{\alpha_j}$

Recall that symmetric polynomials has two different commonly used basis. $\tilde{\sigma}_j(B) = \sum_{j=1}^{k} (\lambda_\alpha)^j$

e.g. $\sigma_1 = \tilde{\sigma}_1 = \lambda_1 + \cdots + \lambda_k$

$\sigma_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{k-1}\lambda_k \qquad \tilde{\sigma}_2 = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_k^2$

$\Rightarrow 2\sigma_2 + \tilde{\sigma}_2 = \sigma_1^2$ See Morita, P. 195

Consider $\tilde{\sigma}_j(F) = \mathrm{tr}(\overbrace{F\wedge\cdots\wedge F}^{j\text{th}})$

$d\tilde{\sigma}_j = \mathrm{tr}(dF\wedge F \wedge\cdots\wedge F) + \mathrm{tr}(F\wedge dF \wedge \cdots \wedge F)$
$\quad + \cdots + \mathrm{tr}(F\wedge\cdots\wedge F \wedge dF)$

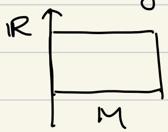
$=$ similar as that in $3° = 0$

5° choice of different connection?

<u>recall</u>. if ∇ and $\tilde{\nabla}$ are two connections on E,

$(1-t)\nabla + t\tilde{\nabla}$ is also a connection on E

<u>idea</u> Regard E as a vector bundle over $M \times \mathbb{R}$ (or $[0,1]$)



$\{\mathcal{U}_\alpha \times \mathbb{R}\}$: open cover for $M \times \mathbb{R}$
(x, t)

transition $g_{\alpha\beta} = g^E_{\alpha\beta}(x)$

Suppose that $\nabla = d + A_\alpha$, $\tilde{\nabla} = d + \tilde{A}_\alpha$ — d on M

Consider $\boldsymbol{\nabla} = d + (1-t)A_\alpha + t\tilde{A}_\alpha$ — d on $M \times \mathbb{R}$

$$\Rightarrow \mathbf{F} = d\big((1-t)A_\alpha + t\tilde{A}_\alpha\big) + \big((1-t)A_\alpha + t\tilde{A}_\alpha\big) \wedge (\,\cdots\,)$$

$$= dt \wedge (\tilde{A}_\alpha - A_\alpha) + (1-t)F_\alpha + t\tilde{F}_\alpha - t(1-t)(A_\alpha - \tilde{A}_\alpha) \wedge (A_\alpha - \tilde{A}_\alpha)$$

$$\mathbf{F}\big|_{t=0} = F, \quad \mathbf{F}\big|_{t=1} = \tilde{F}$$

- recall that $\Lambda^k T^*_{(p,t)}(M \times \mathbb{R}) = (dt \wedge \Lambda^{k-1}_p T^*_p M) \oplus \Lambda^k T^*_p M$
- note that $(\sigma_j(\mathbf{F}))\big|_{t=0} = \sigma_j(F) = \beta(0)$

  $(\sigma_j(\mathbf{F}))\big|_{t=1} = \sigma_j(\tilde{F}) = \beta(1)$

But $\sigma_j(\mathbf{F})$ is a closed $2j$-form on $M \times \mathbb{R}$

$= dt \wedge \alpha(t) + \beta(t)$

$\alpha(t)$: a path in $\Omega^{k-1}(M)$. $\beta(t)$: a path in $\Omega^k(M)$

$$0 = d(\sigma_j(\mathbf{F})) = -dt \wedge d_M\alpha(t) + d_M\beta(t) + dt \wedge \frac{\partial \beta}{\partial t}$$

$$\Rightarrow \frac{\partial \beta}{\partial t} = d_M \alpha(t) \quad \text{and} \quad d_M \beta(t) = 0$$

Thus, $\beta(1) - \beta(0) = \int_0^1 (d_M \alpha(t))\, dt = d_M\left(\int_0^1 \alpha(t)\, dt\right)$

To sum up, $[\sigma_j(F)] = [\sigma_j(\tilde{F})] \in H^{2j}_{dR}(M)$

§II. Chern class

1° <u>def</u> For a complex vector bundle $\mathbb{C}^k \to E \to M$, the total Chern class is defined to be

$$\det\left(\mathbb{I} + \frac{i}{2\pi} F_\nabla\right) = 1 + c_1(E) + c_2(E) + \cdots + c_k(E)$$

F_∇ ↳ curvature of any chose connection on E

2° <u>discussion</u> ① Consider differential forms with $\mathbb{C}$-coefficient
⇝ From last section, $c_j(E) \in H^{2j}_{dR}(M;\mathbb{C})$

② <u>claim</u> In fact, $c_j(E) \in H^{2j}_{dR}(M;\mathbb{R}) \subset H^{2j}_{dR}(M;\mathbb{C})$

Indeed, we can find a representative which is already a real-valued $2j$-form.

Endow E a bundle (Hermitian) metric, and choose a metric connection ∇.

⇝ We can use unitary trivialization, and curvatures are skew-Hermitian 2-forms.

$F_\nabla = -F_\nabla^*$ (on each chart)

$$1 + c_1(E) + \cdots + c_k(E) = \det\left(\mathbb{I} + \frac{i}{2\pi} F_\nabla\right) \qquad \overline{F_\nabla}^* = -\overline{F_\nabla}$$

$$= \det\left(\mathbb{I} + \frac{i}{2\pi} F_\nabla^*\right)$$

$$1 + \overline{c_1(E)} + \cdots + \overline{c_k(E)} = \overline{\det\left(\mathbb{I} + \frac{i}{2\pi} F_\nabla\right)}$$

Hence, $c_j(E) = \overline{c_j(E)}$, and $c_j(E) \in \Omega^{2j}(M;\mathbb{R})$

③ <u>rmk</u> Moreover, $c_j(E) \in \underbrace{H^{2j}(M;\mathbb{Z})}_{\text{singular cohomology}} \otimes \mathbb{R} \subset H^{2j}_{dR}(M;\mathbb{R})$

One can actually define $c_j(E) \in H^{2j}(M;\mathbb{Z})$ by the method of algebraic topology, which does NOT use connection and curvature, and captured the "torsion"

3° If $E_1 \cong E_2$ are isomorphic bundles over M, then $c_j(E_1) = c_j(E_2)$

pf: $\begin{tikzcd} E_1 \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & E_2 \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$

Choose a connection ∇ on E_2

Define a connection $\tilde{\nabla}$ on E_1 by

$$\tilde{\nabla}_X s = \varphi^{-1} \nabla_X(\varphi(s))$$

In terms of local trivializations, $\varphi_u : \mathcal{U} \to Gl(k;\mathbb{C})$

It is not hard to find that $F_{\tilde{\nabla}} = \varphi_u^{-1} F_\nabla \varphi_u$

Therefore, they have the same invariant polynomial ※

4° example ① For $\mathbb{C}^2$-bundle.

$$\det\left(I + \frac{i}{2\pi} F_\nabla\right) = 1 + \frac{i}{2\pi} \operatorname{tr}(F_\nabla) - \frac{1}{4\pi^2} \det(F_\nabla)$$

If ∇ is a metric connection, F_∇ is skew-Hermitian

Let us further assume $\operatorname{tr}(F_\nabla) = 0$ $\Rightarrow c_1(E) = 0$

$$F_\nabla = \begin{bmatrix} i\omega_1 & -\omega_2 + i\omega_3 \\ \omega_2 + i\omega_3 & -i\omega_1 \end{bmatrix}$$ in unitary frame, $\omega_j \in \Omega^2(M;\mathbb{R})$

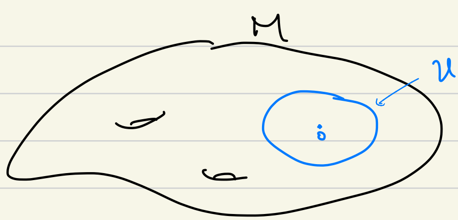
$$\det(F_\nabla) = \sum_{j=1}^{3} \omega_j \wedge \omega_j$$

$$\tilde{\sigma}(F_\nabla) = \operatorname{tr}(F_\nabla \wedge F_\nabla) = \operatorname{tr}\left(\begin{bmatrix} i\omega_1 & -\omega_2 + i\omega_3 \\ \omega_2 + i\omega_3 & -i\omega_1 \end{bmatrix} \wedge \begin{bmatrix} i\omega_1 & -\omega_2 + i\omega_3 \\ \omega_2 + i\omega_3 & -i\omega_1 \end{bmatrix} \right)$$

$$= \operatorname{tr}\left(\begin{bmatrix} -\sum \omega_j^2 & 0 \\ 0 & -\sum \omega_j^2 \end{bmatrix} \right) = -2\det(F_\nabla)$$

$$\Rightarrow c_2(E) = -\frac{1}{4\pi^2} \det(F_\nabla) = \frac{1}{8\pi^2} \operatorname{tr}(F_\nabla \wedge F_\nabla)$$

(ii) M: closed oriented 4-manifold



$\mathcal{U} \cong B(0;1) \subset \mathbb{R}^4$ $\quad \mathcal{V} = M \setminus \{0\}$

$$g : \mathcal{U} \cap \mathcal{V} \to SU(2) \cong S^3$$

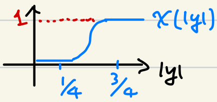
$$x \mapsto \frac{1}{|x|^2} \begin{bmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{bmatrix}$$

For any $m \in \mathbb{Z}$, define E_m by $g_{vu} = g^m$

(ii) construct ∇ on E_m

over $\mathcal{U}$, $\nabla = d + A_u$. over $\mathcal{V}$, $\nabla = d + A_v$

with $A_u = g^{-m} d g^m + g^{-m} A_v g^m$



Use the partition of unity construction.

$\Rightarrow A_u = \chi(|y|) \underbrace{g^{-m} d g^m}_{\text{non-smooth at } 0}$: smooth on $B(0;1)$

$\Rightarrow A_v = g^m \underbrace{(1-\chi)}_{\text{zero when } |z| > \frac{3}{4}\text{, extends by } 0 \text{ on rest of } \mathcal{V}} (g^{-m} d g^m)\, g^{-m}$

(iii) By construction, F_∇ only support on $\mathcal{U}$

Since $\det(g^m) = 1$, $d(\det(g^m)) = 0 = \cancel{\det(g^m)}^{\,1} \operatorname{tr}(g^{-m} d g^m)$

$\boxed{\text{check}}$ A_u and $F = dA_u + A_u \wedge A_u$

are both traceless and skew-Hermitian

Hence, $c_1(E_m) = 0$

(iv) $c_2(E_m) = \,?\, \in H^4_{dR}(M)$. Let us evaluate $\int_M c_2(E_m)$

$$\int_M c_2(E_m) = \frac{1}{8\pi^2} \int_{B(0;1)} \operatorname{tr}(F \wedge F)$$ support & c_2 relation

But $\operatorname{tr}(F \wedge F)$ is a closed 4-form on $B(0;1)$

Poincaré lemma. $\operatorname{tr}(F \wedge F) = d\,(3\text{-form})$

$\boxed{\text{check}}$ $\operatorname{tr}(F \wedge F) = d \operatorname{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$

(See §14.9 in Taubes: Differential Geometry)

Stokes

$$= \frac{1}{8\pi^2} \int_{|x|=1} \operatorname{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

when $|y| = 1$ $A = g^{-m} d g^m$

Denote g^m by h. $A = h^{-1} dh$ $dA = dh^{-1} \wedge dh$

$h h^{-1} = \mathbb{I} \Rightarrow (dh) h^{-1} + h\, dh^{-1} = 0 \Rightarrow dh^{-1} = -h^{-1} (dh) h^{-1}$

$\Rightarrow \operatorname{tr}(A\wedge dA + \frac{2}{3} A\wedge A\wedge A) = \operatorname{tr}(-h^{-1}dh\wedge h^{-1}dh\wedge h^{-1}dh + \frac{2}{3}$ same $)$

Hence. $\displaystyle\int_M c_2(E_m) = \frac{-1}{24\pi^2}\int_{|x|=1} \operatorname{tr}(h^{-1}dh\wedge h^{-1}dh\wedge h^{-1}dh)$

(v) $\{|x|=1\} = S^3$ $\quad \sum_{j=1}^{4} x_j\, dx_j = 0$

when $m=1$. $\quad g^{-1}dg = \begin{bmatrix} x_1 - ix_2 & x_3 - ix_4 \\ -x_3 - ix_4 & x_1 + ix_2 \end{bmatrix}\begin{bmatrix} dx_1 + i dx_2 & -dx_3 + i dx_4 \\ dx_3 + i dx_4 & dx_1 - i dx_2 \end{bmatrix}$

$$= \begin{bmatrix} i(-x_2dx_1 + x_1dx_2 - x_4dx_3 + x_3dx_4) & \cdots \\ (-x_3dx_1 + x_4dx_2 + x_1dx_3 - x_2dx_4) & \\ +i(-x_4dx_1 - x_3dx_2 + x_2dx_3 + x_1dx_4) & \cdots \end{bmatrix} = \begin{bmatrix} i\mu_1 & -\mu_2 + i\mu_3 \\ \mu_2 + i\mu_3 & -i\mu_1 \end{bmatrix}$$

$$\operatorname{tr}(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg) = \operatorname{tr}\left(\begin{bmatrix} i\mu_1 & -\mu_2 + i\mu_3 \\ \mu_2 + i\mu_3 & -i\mu_1 \end{bmatrix}\wedge\begin{bmatrix} i\mu_1 & -\mu_2 + i\mu_3 \\ \mu_2 + i\mu_3 & -i\mu_1 \end{bmatrix}\wedge \cdots\right)$$

$$= \operatorname{tr}\left(\begin{bmatrix} -2i\mu_2\wedge\mu_3 & +2\mu_3\wedge\mu_1 - 2i\mu_1\wedge\mu_2 \\ -2\mu_3\wedge\mu_1 - 2i\mu_1\wedge\mu_2 & 2i\mu_2\wedge\mu_3 \end{bmatrix}\wedge\begin{bmatrix} i\mu_1 & -\mu_2 + i\mu_3 \\ \mu_2 + i\mu_3 & -i\mu_1 \end{bmatrix}\right)$$

$$= \operatorname{tr}\left(\begin{bmatrix} 6\mu_1\wedge\mu_2\wedge\mu_3 & \cdots \\ \cdots & 6\mu_1\wedge\mu_2\wedge\mu_3 \end{bmatrix}\right) = 12\,\mu_1\wedge\mu_2\wedge\mu_3$$

But μ_1, μ_2, μ_3 : everywhere orthonormal, positive oriented

$\Rightarrow \displaystyle\int_{S^3} \mu_1\wedge\mu_2\wedge\mu_3 = 2\pi^2$ (or do direct integration).

at $(1,0,0,0) \rightsquigarrow dx^2\wedge dx^3\wedge dx^4$

outer normal $= \frac{\partial}{\partial x^1}$

Hence, $\int c_2(E_1) = -1$.

(vi) $c_2(E_m)$ <u>quick reason</u>

$\frac{1}{24\pi^2}\operatorname{tr}(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg) \in \Omega^3(S^3)$ total integration -1

$\frac{1}{24\pi^2}\operatorname{tr}(h^{-1}dh\wedge h^{-1}dh\wedge h^{-1}dh)$: pull-back of the volume form

by $S^3 \longrightarrow S^3 \cong SU(2)$
$\quad g \longmapsto h = g^m$

(g : normal $\Rightarrow g = U^*\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}U \quad U^{-1} = U^*$
$\Rightarrow h = U^*\begin{bmatrix}\lambda_1^m & 0 \\ 0 & \lambda_2^m\end{bmatrix}U \Rightarrow U$: eigenbasis
for "generic" h, it has exactly m roots in $SU(2)$)

Hence, $\int c_2(E_m) = -m$ (one can also work harder to simplify the 3-form and evaluate it)

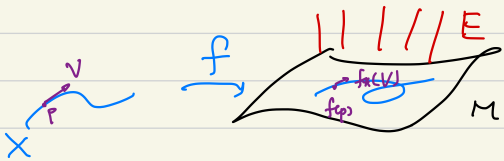
Similar construction for $\mathbb{R}^k$-bundle $\rightsquigarrow$ Pontryagin class in $\boxed{\text{Homework}}$

§ III. aside : pull-back connection

setting: $E \to M$ with connection ∇. $f : X \to M$ smooth

goal: define $f^*\nabla$ on f^*E

1° general criterion. $f^*E = \{ (x, e) \in X \times E \mid f(x) = \pi(e) \in M \}$



$f^*\nabla$ is the connection determined by $(f^*\nabla)_v (f^*s) = f^*(\nabla_{f_* v} s)$ $\forall\, v \in TX$, $s \in \Gamma(E)$

2° explicit construction.

$E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$ with $g_{\alpha\beta}$, $\nabla|_{U_\alpha} = d + A_\alpha$

$V_\alpha = f^{-1}(U_\alpha)$ f^*E is given by $g_{\alpha\beta} \circ f$

Define $f^*\nabla$ by $f^*\nabla|_{V_\alpha} = d + f^*(A_\alpha)$

$$\begin{array}{ccccc}
A_\beta \in \Omega^1(U_\beta; \mathbb{R}^{k\times k}) & \xrightarrow{g_{\alpha\beta}} & \Omega^1(U_\alpha; \mathbb{R}^{k\times k}) \ni A_\alpha \\
\downarrow f^* & \circlearrowright & \downarrow f^* \\
f^*A_\beta \in \Omega^1(V_\beta; \mathbb{R}^{k\times k}) & \xrightarrow{f^* g_{\alpha\beta}} & \Omega^1(V_\alpha; \mathbb{R}^{k\times k}) \ni f^*A_\alpha
\end{array}$$

... by chain rule ...