§I. brief review

1° connection (an equivalent formulation) is a map

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E) \quad \text{such that}$$

$$\underset{\circ}{\nabla}(fs) = df \otimes s + f \cdot \underset{\circ}{\nabla} s$$

the T^*M-part where you can plug in a vector field

2° in terms of defn 2, E is given by $\{\mathcal{U}_\alpha\}$ and

$g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(k;\mathbb{R})$ satisfying the cocycle condition

A connection consists of A_α: $k\times k$-matrix of 1-forms on each $\mathcal{U}_\alpha$

such that $A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$

$$\left(\begin{aligned} s_\alpha &= g_{(\alpha\beta)} s_\beta \\ (ds_\alpha + A_\alpha s_\alpha) &= d(g s_\beta) + A_\alpha g s_\beta \\ \Rightarrow A_\beta &= g^{-1} dg + g^{-1} A_\alpha g \end{aligned} \right)$$

§II. commuting derivatives

1° motivation $\partial_j \partial_i f = \partial_i \partial_j f$

What happens if $f \in \Gamma(E)$, $\partial_i \rightsquigarrow \nabla_{\partial_i}$?

setting $E \to M$ vector bundle with a connection ∇

$\mathcal{U} \subset M$ coordinate neighborhood and $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^k$

With the trivialization, $\nabla = d + \boxed{A} = \sum_i A_i \, dx^i$

each $A_i : \mathcal{U} \to M(k\times k;\mathbb{R})$

$$\begin{aligned} \nabla_{\partial_i}\nabla_{\partial_j} s - \nabla_{\partial_j}\nabla_{\partial_i} s &= (\partial_i + A_i)(\partial_j s + A_j s) - (\partial_j + A_j)(\partial_i s + A_i s) \\ &= \partial_i\partial_j s + (\partial_i A_j) s + \cancel{A_j(\partial_i s)} + \cancel{A_i(\partial_j s)} + A_i A_j s \\ &\quad - \partial_j \partial_i s - (\partial_j A_i) s - \cancel{A_i(\partial_j s)} - \cancel{A_j(\partial_i s)} - A_j A_i s \\ &= (\partial_i A_j - \partial_j A_i + [A_i, A_j])\, s \quad \text{———} \quad (☆) \end{aligned}$$

2° **observation** commuting derivative needs not be zero.

However, it has no derivative on $\mathfrak{X}(E)$

defn Define the curvature of ∇ to be the tri-linear map

$$\mathfrak{X}(M)\times\mathfrak{X}(M)\times\Gamma(E)\longrightarrow\Gamma(E)$$

$$(X,\ Y,\ s)\longmapsto\nabla_X\nabla_Y s-\nabla_Y\nabla_X s-\nabla_{[X,Y]}s$$

prop It is $C^\infty(M)$-linear in each argument

pf:

$$(X,Y,fs)\longmapsto\nabla_X\nabla_Y(fs)-\nabla_Y\nabla_X(fs)-\nabla_{[X,Y]}(fs)$$

$$=\nabla_X(Y(f)s+f\nabla_Y s)-\nabla_Y(X(f)s+f\nabla_X s)$$

$$-[X,Y](f)\cdot s-f\nabla_{[X,Y]}s$$

$$=\cancel{X(Y(f))}s+\cancel{Y(f)\nabla_X s}+\cancel{X(f)\nabla_Y s}+f\nabla_X\nabla_Y s$$

$$-\cancel{Y(X(f))}s-\cancel{X(f)\nabla_Y s}-\cancel{Y(f)\nabla_X s}-f\nabla_Y\nabla_X s$$

$$-(\cancel{X(Y(f))}-\cancel{Y(X(f))})\cdot s-f\nabla_{[X,Y]}s$$

For (fX,Y,s):

$$\nabla_{fX}\nabla_Y s-\nabla_Y\nabla_{fX}s-\nabla_{[fX,Y]}s$$

$$=f\nabla_X\nabla_Y s-\cancel{Y(f)\nabla_X s}-f\nabla_Y\nabla_X s$$

$$-f\nabla_{[X,Y]}s+\cancel{Y(f)\nabla_X s}$$

$$[fX,Y]$$
$$=fX(Y)-Y(fX)$$
$$=f[X,Y]-Y(f)X$$

Similar computation for $(X,\ fY,\ s)$ ※

rmk For ∂_i,∂_j. $[\partial_i,\partial_j]=0$

3° **discussion** Due to the prop last time, curvature is a tensor. More precisely, it is a section of $(\Lambda^2T^*M)\otimes\mathrm{End}(E)$

↪ skew-symmetric in (X,Y)

Usually denote it by $\sum_{i,j} \frac{1}{2} \underbrace{F_{ij}}_{\hookrightarrow \text{End}(E) \text{ or } k\times k\text{-matrix}} dx^i \wedge dx^j$

where $F_{ij} = \partial_i A_j - \partial_j A_i - [A_i, A_j] = -F_{ji}$ see (✡)

Note that $\left(\sum_{k,l} \frac{1}{2} F_{kl}\, dx^k \wedge dx^l\right)(\partial_i, \partial_j) = \frac{1}{2} F_{ij} - \frac{1}{2} F_{ji} = F_{ij}$

rmk Instead of writing End(E) as a k^2 column vector, we usually write it as a $k\times k$ matrix.

If $g_{\alpha\beta}$ is the transition of E, the transition of End(E) goes as follows: $g_{\alpha\beta}^{-1} F_\alpha\, g_{\alpha\beta} = F_\beta$

$$\left(\begin{tikzcd} U_\beta \times \mathbb{R}^k \arrow[r, "g_{\alpha\beta}"] \arrow[d, "F_\beta"] & U_\alpha \times \mathbb{R}^k \arrow[d, "F_\alpha"] \\ U_\beta \times \mathbb{R}^k \arrow[r, "g_{\alpha\beta}"] & U_\alpha \times \mathbb{R}^k \end{tikzcd}\right)$$

$\boxed{\text{HW}}$ One can check that F_{ij} satisfies the above transition rule by using the transition formula of the connection 1-form, A

rmk It is $dA + A \wedge A = d(A_i\, dx^i) + (A_i\, dx^i) \wedge (A_j\, dx^j)$

$$= \frac{\partial A_i}{\partial x_j} dx^j \wedge dx^i + A_i A_j\, dx^i \wedge dx^j$$

$$= \frac{1}{2} \frac{\partial A_j}{\partial x_i} dx^i \wedge dx^j - \frac{1}{2} \frac{\partial A_i}{\partial x_j} dx^i \wedge dx^j$$

$$+ \frac{1}{2} A_i A_j\, dx^i \wedge dx^j - \frac{1}{2} A_j A_i\, dx^i \wedge dx^j = \frac{1}{2} F_{ij}\, dx^i \wedge dx^j$$

§III basic example

Consider TS^2 with the ∇ given by $S^2 \subset \mathbb{R}^3$ as the sphere of radius R. Choose spherical coordinate $X = R(\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi)$

(Note that $N = X/R$)

$X_\phi = R(\cos\phi \cos\theta, \cos\phi \sin\theta, -\sin\phi)$

$X_\theta = R(-\sin\phi \sin\theta, \sin\phi \cos\theta, 0) = R \sin\phi\, (-\sin\theta, \cos\theta, 0)$

$|X_\phi|^2 = R^2$, $|X_\theta|^2 = R^2 \sin^2\phi$, $X_\phi \perp X_\theta$

$$\nabla V = (dV)^T$$

$$d\mathbb{X}_\phi = R(-\sin\phi\cos\theta, -\sin\phi\sin\theta, -\cos\phi)\,d\phi + R(-\cos\phi\sin\theta, \cos\phi\cos\theta, 0)\,d\theta$$

$$\Rightarrow \nabla \mathbb{X}_\phi = \cos\phi/\sin\phi\ d\theta\ \mathbb{X}_\theta$$

$$d\mathbb{X}_\theta = R\cos\phi\,(-\sin\theta, \cos\theta, 0)\,d\phi + R\sin\phi\,(-\cos\theta, -\sin\theta, 0)\,d\theta$$

$$\Rightarrow \nabla \mathbb{X}_\theta = -\sin\phi\cos\phi\ d\theta\ \mathbb{X}_\phi + \cos\phi/\sin\phi\ d\phi\ \mathbb{X}_\theta$$

In terms of the matrix notation.

$$\nabla = d + \boxed{\begin{bmatrix} 0 & -\sin\phi\cos\phi\,d\theta \\ \frac{\cos\phi}{\sin\phi}d\theta & \frac{\cos\phi}{\sin\phi}d\phi \end{bmatrix}} = A$$

$$\text{curvature} = dA + A\wedge A$$

$$= \begin{bmatrix} 0 & (\sin^2\phi - \cos^2\phi) \\ -\frac{1}{\sin^2\phi} & 0 \end{bmatrix} d\phi\wedge d\theta + \underbrace{\begin{bmatrix} 0 & -\sin\phi\cos\phi\,d\theta \\ \frac{\cos\phi}{\sin\phi}d\theta & \frac{\cos\phi}{\sin\phi}d\phi \end{bmatrix} \wedge \begin{bmatrix} 0 & -\sin\phi\cos\phi\,d\theta \\ \frac{\cos\phi}{\sin\phi}d\theta & \frac{\cos\phi}{\sin\phi}d\phi \end{bmatrix}}_{\parallel\ \begin{bmatrix} 0 & \cos^2\phi\ d\phi\wedge d\theta \\ \frac{\cos^2\phi}{\sin^2\phi}\,d\phi\wedge d\theta & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & \sin^2\phi\ d\phi\wedge d\theta \\ -d\phi\wedge d\theta & 0 \end{bmatrix}$$

How does it related to Gaussian curvature?

$$dA = R^2\sin\phi\ d\phi\wedge d\theta$$

$$\begin{cases} F\cdot\mathbb{X}_\phi = -d\phi\wedge d\theta\cdot\mathbb{X}_\theta \\ F\cdot\mathbb{X}_\theta = \sin^2\phi\ d\phi\wedge d\theta\cdot\mathbb{X}_\phi \end{cases}$$

$$\Rightarrow \begin{cases} F\cdot\frac{\mathbb{X}_\phi}{R} = -\frac{1}{R}\,d\phi\wedge d\theta\cdot R\sin\phi\,\frac{\mathbb{X}_\theta}{R\sin\phi} = -\frac{dA}{R^2}\,\frac{\mathbb{X}_\theta}{R\sin\phi} \\ F\cdot\frac{\mathbb{X}_\theta}{R\sin\phi} = \frac{1}{R\sin\phi}\sin^2\phi\ d\phi\wedge d\theta\cdot R\,\frac{\mathbb{X}_\phi}{R} = \frac{dA}{R^2}\,\frac{\mathbb{X}_\phi}{R} \end{cases}$$

In terms of the orthonormal trivializing sections, $\left\{\frac{\mathbb{X}_\phi}{R}, \frac{\mathbb{X}_\theta}{R\sin\phi}\right\}$

$$F = \begin{bmatrix} 0 & K \\ -K & 0 \end{bmatrix} dA$$

Lem If ∇ is a metric connection, and we chose an orthonormal trivialization, then F is skew-symmetric

pf: $A^T = -A$ by $\boxed{HW}$

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + A_iA_j - A_jA_i$$

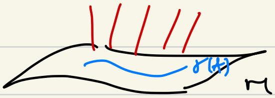
$$F_{ij}^T = -\frac{\partial A_j}{\partial x_i} + \frac{\partial A_i}{\partial x_j} + A_jA_i - A_iA_j = F_{ji} = -F_{ij} \qquad \#$$

§IV parallel transport.

$E \to M$ with ∇

1° $\gamma(t)$: curve on M. $\gamma(0) = p$, $\gamma(1) = q$

∇ gives a way to identify E_p with E_q



defn a section $s(t)$ of $E|_{\gamma(t)}$ is called parallel if $\nabla_{\gamma'(t)} s(t) \equiv 0$

Locally, $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^k$, section $\cong \{\mathbb{R}^n\text{-valued function}\}$

$$\nabla = d + A$$

Parallel equation on $s = (s^1, \dots, s^k)$ reads

$$\frac{ds}{dt} = -\underbrace{A}_{\text{at } \gamma(t)}(\gamma'(t)) \cdot s \qquad \text{———} \quad (P)$$

By ODE, locally solvable for any $s(0)$

↪ then solvable over any $\gamma : [0,1] \to M$

Hence, parallel transport along γ defines $\mathrm{Hom}(E_p, E_q)$

By ODE, it has an inverse, which is given by parallel transport along "$-\gamma$"

2° from a global viewpoint

$$s(t) : \mathcal{U} \to \mathcal{U} \times \mathbb{R}^k$$
$$t \mapsto (\underbrace{\gamma(t)}_{\pi \circ s = \gamma}, \underbrace{s(t)}_{\text{given by } (P)})$$

Let $((x^1, \cdots, x^n), (w^1, \cdots, w^k))$ be the coordinate for $\mathcal{U} \times \mathbb{R}^k$

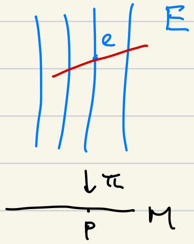
(P) means $\frac{\partial w^\mu}{\partial t} + \sum_{\nu=1}^{k}\sum_{i=1}^{n} A^\mu_{\nu i}(x) \frac{\partial x^i}{\partial t} w^\nu = 0$

$\Longleftrightarrow$ $dw^\mu + A^\mu_\nu w^\nu$: 1-form on $\mathcal{U} \times \mathbb{R}^k$ vanishes on $s(t)$

kernel of $dw^\mu + A^\mu_\nu w^\nu$? k 1-forms on $\mathcal{U} \times \mathbb{R}^k$

nullity is n in general

due to its special form. kernel $= \left\{ \frac{\partial}{\partial x^i} - A^\mu_\nu\left(\frac{\partial}{\partial x^i}\right) w^\nu \frac{\partial}{\partial w^\mu} \right\}_{i=1}^{n}$



$\ker \pi_*|_e \cong E_p$ ($\pi(E_p) = p$. and E_p admits a vector space structure)

upshot a connection gives an injective linear map

$$\begin{tikzcd} T_pM \arrow[r, "\mathcal{L}_e"] \arrow[dr, "\mathbb{I}"'] & T_eE \arrow[d, "\pi_*"] \arrow[dl, phantom, "\circlearrowleft"] \\ & T_pM \end{tikzcd}$$

such that

defn The above map is called the horizontal lifting given by the connection

$\mathcal{H} = \left\{ \coprod_{e \in E} \mathcal{L}_e(T_pM) \right\}$ is a rank n distribution of TE

recap parallel transport $\Longleftrightarrow$ $\begin{cases} \pi \circ s(t) = \gamma(t) \\ \text{and } s'(t) \in \mathcal{H}_{s(t)} \end{cases}$

3° integrability of $\mathcal{H}$ (in the sense of Frobenius theorem)

We use the "dual version". which is explained in the TA session last semester

$d(dw^\mu + A^\mu_\nu w^\nu) \overset{?}{\in}$ ideal $\{dw^\mu + A^\mu_\nu w^\nu\}_{\mu=1}^{k}$

$= dA^\mu_\nu\ w^\nu - A^\mu_\nu \wedge dw^\nu$

$= dA^\mu_\delta\ w^\delta - A^\mu_\nu \wedge (dw^\nu + A^\nu_\delta w^\delta - A^\nu_\delta w^\delta)$

$= \underbrace{(dA^\mu_\delta + A^\mu_\nu \wedge A^\nu_\delta)}_{\text{No } dw} w^\delta - A^\mu_\nu \wedge \overbrace{(dw^\nu + A^\nu_\delta w^\delta)}^{\text{o.K.}}$

Hence, involutive $\Longleftrightarrow$ $F \equiv 0$

4° holonomy (和樂) and curvature

← provide another geometric meaning. we will not do the proof in this class

γ : closed curve on M

$\gamma(0) = \gamma(1) = p$

Parallel transport along γ defines an automorphism of E_p

$$\Rightarrow \rho : \{\text{closed curves at } p\} \to \overset{GL(k;\mathbb{R})}{\overset{\wr\wr}{\mathrm{Aut}(E_p)}}$$
$$\gamma \mapsto \rho_\gamma$$

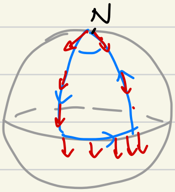
← holonomy around γ

fact γ_s : family of closed curves at p

$$\mathrm{End}(E_p) \ni \frac{d}{ds}\Big|_{s=0} \rho_{\gamma_s} = \pm \int_{\gamma_0} F\left(\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right) dt$$

(Ambrose-Singer theorem)

eg



γ = blue curve

ρ_γ = rotation on $T_N S^2$

↳ by angle $\sim$ $\iint K \, d(\text{area})$

§V. extract information from curvature

$F \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(E))$ ← k×k matrices of 2-forms

For different trivializations, $g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta} = F_\beta$

For instance, $\mathrm{tr}(F_\alpha) = \mathrm{tr}(F_\beta)$

Hence, $\mathrm{tr}(F) \in \Omega^2(TM)$ is well-defined

It depends on ∇, but not the trivialization

1° a polynomial $f : M(k\times k;\mathbb{R}) \cong \mathbb{R}^{k^2} \to \mathbb{R}$ is called an *invariant polynomial* if $f(g^{-1}Bg) = f(B)$
$\forall\, B \in M(k\times k;\mathbb{R}),\ g \in GL(k;\mathbb{R})$

lemma Write $\det(\mathbb{I}+sB) = 1 + s\,\sigma_1(B) + \cdots + s^k\sigma_k(B)$

each $\sigma_j(B)$ is an invariant polynomial of deg j

Note that $\sigma_1(B) = \operatorname{tr}(B)$. $\sigma_k(B) = \det(B)$

pf: linear algebra #

fact These are all invariant polynomials. (but we will not need this fact)

2° dF ? $\qquad \nabla = d + A$

$$F = dA + A\wedge A$$

$$\Rightarrow dF = \overset{0}{\cancel{d^2A}} + dA\wedge A - A\wedge dA$$

$$= (F - A\wedge A)\wedge A - A\wedge(F - A\wedge A)$$

$$\Rightarrow dF = F\wedge A - A\wedge F$$ this is called the Bianchi identity

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next week ?

3° $\det(\mathbb{I}+sF) = 1 + s\,\sigma_1(F) + \cdots + s^k\sigma_k(F)$

$$(\mathbb{I}+sF)^{-1} = \mathbb{I} - sF + s^2F^2 \mp \cdots + \underbrace{(-1)^j s^j F^j}_{\text{vanishes after } j > \dim M/2} + \cdots$$

recall:

$d(\det B) = (\det B)^{-1}\operatorname{tr}(B^{-1}dB)$ by Cramer's rule

$$\Rightarrow d(\det(\mathbb{I}+sF))$$

$$= (\det(\mathbb{I}+sF))^{-1}\operatorname{tr}\left((\mathbb{I}+sF)^{-1}\wedge d(\mathbb{I}+sF)\right)$$

$$= (\det(\mathbb{I}+sF))^{-1}\underbrace{\operatorname{tr}\left((\mathbb{I} - sF + s^2F^2 \mp \cdots + (-1)^j s^j F^j)\wedge s(F\wedge A - A\wedge F)\right)}_{\sum_j (-1)^{j+1}s^{j+1}\underbrace{\operatorname{tr}(F^j\wedge(F\wedge A - A\wedge F))}_{\operatorname{tr}(F^{j+1}\wedge A) - \operatorname{tr}(F^j\wedge A\wedge F)}}$$

F: 2-forms

A: 1-form

$$= \operatorname{tr}(F^{j+1}\wedge A) - \operatorname{tr}(F\wedge F^j\wedge A) = 0$$

Therefore, $\sigma_j(F) = 0 \quad \forall j$.

4° sketch of another proof

Suppose that $B = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}$ $\Rightarrow \sigma_j(B) = \sum_{\alpha_1<\cdots<\alpha_j} \lambda_{\alpha_1}\lambda_{\alpha_2}\cdots\lambda_{\alpha_j}$

Recall that symmetric polynomials has two different commonly used basis. $\tilde{\sigma}_j(B) = \sum_{j=1}^{k} (\lambda_\alpha)^j$

e.g. $\sigma_1 = \tilde{\sigma}_1 = \lambda_1 + \cdots + \lambda_k$

$\sigma_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{k-1}\lambda_k$ $\quad \tilde{\sigma}_2 = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_k^2$

$\Rightarrow 2\sigma_2 + \tilde{\sigma}_2 = \sigma_1^2$ See Morita, P.195

Consider $\tilde{\sigma}_j(F) = tr(\overbrace{F\wedge\cdots\wedge F}^{j\text{th}})$

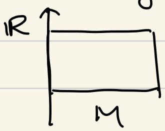
$$d\tilde{\sigma}_j = tr(dF\wedge F\wedge\cdots\wedge F) + tr(F\wedge dF\wedge\cdots\wedge F) + \cdots + tr(F\wedge\cdots\wedge F\wedge dF)$$

$= $ similar as that in 3° $= 0$

5° choice of different connection?

recall. if ∇ and $\tilde{\nabla}$ are two connections on E, $(1-t)\nabla + t\tilde{\nabla}$ is also a connection on E

idea Regard E as a vector bundle over $M\times\mathbb{R}$ (or $[0,1]$)



$\{\mathcal{U}_\alpha \times \mathbb{R}\}$: open cover for $M\times\mathbb{R}$

(x, t) transition $g_{\alpha\beta} = g^{E}_{\alpha\beta}(x)$

Suppose that $\nabla = d + A_\alpha$, $\tilde{\nabla} = d + \tilde{A}_\alpha$ $\quad$ d on M

Consider $\boldsymbol{\nabla} = d + (1-t)A_\alpha + t\tilde{A}_\alpha$ $\quad$ d on $M\times\mathbb{R}$

$$\Rightarrow \mathbf{F} = d((1-t)A_\alpha + t\tilde{A}_\alpha) + ((1-t)A_\alpha + t\tilde{A}_\alpha)\wedge(\cdots)$$

$$= dt\wedge(\tilde{A}_\alpha - A_\alpha) + (1-t)F_\alpha + t\tilde{F}_\alpha - t(1-t)(A_\alpha - \tilde{A}_\alpha)\wedge(A_\alpha - \tilde{A}_\alpha)$$

$\mathbf{F}|_{t=0} = F$, $\mathbf{F}|_{t=1} = \tilde{F}$

- recall that $\Lambda^k T^*_{(p,t)}(M \times \mathbb{R}) = (dt \wedge \Lambda^{k-1}_p T^*_p M) \oplus \Lambda^k T^*_p M$
- note that $(\sigma_j(\mathbf{F}))|_{t=0} = \sigma_j(F) = \beta(0)$

$(\sigma_j(\mathbf{F}))|_{t=1} = \sigma_j(\tilde{F}) = \beta(1)$

But $\sigma_j(\mathbf{F})$ is a closed $2j$-form on $M \times \mathbb{R}$

$= dt \wedge \alpha(t) + \beta(t)$

$\alpha(t)$: a path in $\Omega^{k-1}(M)$. $\beta(t)$: a path in $\Omega^k(M)$

$$0 = d(\sigma_j(\mathbf{F})) = -dt \wedge d_M \alpha(t) + d_M \beta(t) + dt \wedge \frac{\partial \beta}{\partial t}$$

$$\Rightarrow \frac{\partial \beta}{\partial t} = d_M \alpha(t) \quad \text{and} \quad d_M \beta(t) = 0$$

Thus, $\beta(1) - \beta(0) = \int_0^1 (d_M \alpha(t))\, dt = d_M \left(\int_0^1 \alpha(t)\, dt \right)$

To sum up, $[\sigma_j(F)] = [\sigma_j(\tilde{F})] \in H^{2j}_{dR}(M)$