§I. taking derivative of a section?

$E \to M$: vector bundle , $s: M \to E$ a section

$\boxed{\text{goal}}$ the derivative of s ?

0° $s_*: TM \to TE \Rightarrow v \in T_pM$, $s_*(v) \in T_{s(p)}E$

vector space of $\dim = \dim M + \operatorname{rank} E$

Define the derivative so that "$\frac{\partial s}{\partial v}$" $\in E_p$?

1° locally $E|_{\mathcal{U}} \overset{\varphi}{\cong} \underset{(x,\ w)}{\mathcal{U} \times \mathbb{R}^k}$ (Suppose $\mathcal{U}$ is a coordinat chart)

Then, a section is a smooth function $\alpha: \mathcal{U} \to \mathbb{R}^k$

Its partial / directional derivative is still $\mathbb{R}^k$-valued.

e.g. $\frac{\partial \alpha}{\partial x_j}$: still $\mathcal{U} \to \mathbb{R}^k$, hence local section

2° well-defined? If we chose a different trivialization

$E|_{\mathcal{U}} \overset{\tilde{\varphi}}{\cong} \underset{(x,\ \tilde{w})}{\mathcal{U} \times \mathbb{R}^k}$ $\quad \tilde{w} = g(x)w \quad g: \mathcal{U} \to GL(k;\mathbb{R})$

In terms of $\tilde{\varphi}$, α corresponds to $g \cdot \alpha$

$g \cdot (-)$

Its partial derivative is $\frac{\partial}{\partial x_j}(g \cdot \alpha) = g \cdot \frac{\partial \alpha}{\partial x_j} + \frac{\partial g}{\partial x_j} \alpha$

3° ⚠ taking partial derivative does NOT commute with change of the basis.

$\Rightarrow$ The above procedure does NOT define a differential operator.

Another viewpoint : derivative = limit of difference quotient

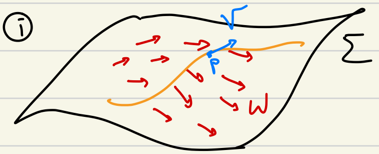
Although each E_p is a vector space, there is no canonical way to identify E_p with E_q for $p \neq q$.

§II two examples

1° $\Sigma^2 \subset \mathbb{R}^3$ (oriented) regular surface

Consider $\mathbb{R}^2 \to T\Sigma \to \Sigma$, section = tangent vector field

(i)



W: tangent vector field. ($\langle W, N\rangle \equiv 0$)

$p \in \Sigma$, $V \in T_p\Sigma$

The derivative of W in V ?

(Follow the procedure of directional derivative)

Choose a curve $\gamma : (-1,1) \to \Sigma$ such that

$$\gamma(0) = p, \quad \gamma'(0) = V \quad \text{———} \quad (\star)$$

$\rightsquigarrow W\circ\gamma : (-1,1) \to \mathbb{R}^3$. Take $\frac{d}{dt}\Big|_{t=0}(W\circ\gamma) \in \mathbb{R}^3$

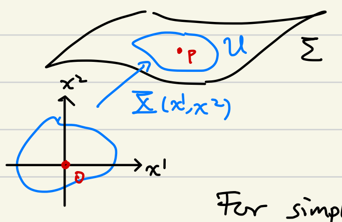
Nothing to do with trivializing $T\Sigma$; it comes from $\Sigma \subset \mathbb{R}^3$

But it needs not to tangent to Σ.

Resolve this issue by taking the orthogonal projection, $\left(\frac{d}{dt}\Big|_{t=0}(W\circ\gamma)\right)^T$

rmk From calculus (chain rule), $\frac{d}{dt}\Big|_{t=0}(W\circ\gamma)$ is independent of the choice of γ. It only depends on $(\star)$.

(ii) coordinate and trivialization



$$\mathcal{U} \times \mathbb{R}^2 \longrightarrow T\Sigma|_{\mathcal{U}}$$

$$((x^1,x^2),(v^1,v^2)) \longmapsto \text{the vector } \sum_j v^j \frac{\partial X}{\partial x^j} \text{ at } X(x^1,x^2)$$

$$\text{vector field } W = \sum_j w^j(x^1,x^2)\frac{\partial X}{\partial x^j}$$

For simplicity, choose $V \in T_p\Sigma$ to be $\frac{\partial X}{\partial x^k}$ $(k = 1 \sim 2)$

Take $\gamma = X(t,0)$ or $X(0,t)$

$$\Rightarrow W\circ\gamma = \sum_j w^j(t,0)\frac{\partial X}{\partial x^j}\Big|_{(t,0)}$$

$$\Rightarrow \frac{d}{dt}(W\circ\gamma) = \sum_j \left(\underbrace{\frac{dw^j(t,0)}{dt}}_{=\frac{\partial w^j}{\partial x^1}}\frac{\partial X}{\partial x^j} + w^j(t,0)\frac{\partial^2 X}{\partial x^1 \partial x^j}\right)$$

$$\Rightarrow \left(\frac{d}{dt}(W\circ\gamma)\right)^T = \sum_j \frac{\partial w^j}{\partial x^1}\frac{\partial X}{\partial x^j} + w^j\left(\frac{\partial^2 X}{\partial x^1 \partial x^j}\right)^T = \sum_k \Gamma^k_{1j}\frac{\partial X}{\partial x^k}$$

denote the coeffient by Γ^k_{1j}

<u>Upshot</u> The $\frac{\partial \Sigma}{\partial x^j}$ component of the "derivative" of W in $\frac{\partial \Sigma}{\partial x^i}$ is $\frac{\partial w^j}{\partial x^i} + \Gamma^j_{ik} w^k$.

<u>summary</u> in terms of $T\Sigma \cong \mathcal{U} \times \mathbb{R}^2$ (induced from coordinate chart)

$W = (w^1, w^2)$ "derivative" in $\sum_i v^i \frac{\partial}{\partial x^i}$ is

$$\sum_i v^i \left(\frac{\partial w^1}{\partial x^i} + \Gamma^1_{ij} w^j,\ \frac{\partial w^2}{\partial x^i} + \Gamma^2_{ij} w^j \right)$$

<u>rmk</u> $\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial}{\partial x^i} g_{lj} + \frac{\partial}{\partial x^j} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \right)$ from last semester.

<u>lesson</u> It is a well-defined derivative. (with the help of $\Sigma \subset \mathbb{R}^3$). In terms of local trivialization, the "derivative" consists of a zeroth order part.

Compare with ☹ in §I. $\rightsquigarrow$ use zeroth order part to absorb $\frac{\partial g}{\partial x^j} \alpha$.

2° tautological (complex) line bundle over $\mathbb{CP}^1 \cong S^2$

<u>recall</u> The tautological line bundle by definition is a subbundle of the trivial bundle $\underline{\mathbb{C}}^2 = \mathbb{CP}^1 \times \mathbb{C}^2$.

(i) $E = \{ ([L], \xi) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid \xi \,/\!/\, L \}$

$\mathbb{CP}^1 = \underset{[z:1]}{\mathcal{U}} \cup \underset{[1:w]}{\mathcal{V}}$ both $\mathcal{U}, \mathcal{V} \cong \mathbb{C}$ and $\mathcal{U} \cap \mathcal{V} = \mathbb{C}^*$ with $zw = 1$

Consider $\mathcal{U} \times \mathbb{C} \cong E|_{\mathcal{U}}$

$(z, u) \mapsto ([z:1], (uz, u))$ $\Rightarrow$ $v = uz$

and $\mathcal{V} \times \mathbb{C} \cong E|_{\mathcal{V}}$

$(w, v) \mapsto ([1:w], (v, vw))$

$g_{vu} = z$

$g_{uv} = z^{-1} = w$

just a notation to remind us that they need not to be "holomorphic"

A section is given by $u = u(z, \bar{z})$, $v = v(w, \bar{w})$

satisfying $v = z \cdot u$

(ii) Similar as example I°, although u & v are not well-defined function on $\mathbb{CP}^1$, $(uz, u) = (v, vw)$ is a well-defined $\mathbb{C}^2$-valued function on $\mathbb{CP}^1$

We can take the differential, and do orthogonal projection onto $E_{[L]}$ at each $[L] \in \mathbb{CP}^1$.

with respect to the standard Hermitian metric on $\mathbb{C}^2$

- $d(uz, u) = \big(\underbrace{(du)}_{\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z}} z + u\,dz,\ du \big)$

$E|_{[z:1]} \,/\!/\, (z, 1)\big/\sqrt{1+|z|^2}$ orthogonal $= (-1, \bar{z})\big/\sqrt{1+|z|^2} =: N$

- $d(uz, u) - \langle d(uz, u), N \rangle N$

$$= (z\,du + u\,dz,\ du) - \frac{1}{1+|z|^2}\left(\cancel{-z\,du} - u\,dz + \cancel{z\,du}\right)(-1, \bar{z})$$

$$= \left(z\,du + \frac{|z|^2}{1+|z|^2} u\,dz,\ du + \frac{\bar{z}}{1+|z|^2} u\,dz \right)$$

$$= \left(du + \frac{\bar{z}}{1+|z|^2} u\,dz\right)(z, 1)$$

(iii) Hence. $u \longmapsto du + \frac{\bar{z}}{1+|z|^2} u\,dz$

That is to say, the derivative in $\frac{\partial}{\partial x}$ is $\frac{\partial u}{\partial x} + \frac{\bar{z}}{1+|z|^2} u$

in $\frac{\partial}{\partial y}$ is $\frac{\partial u}{\partial y} + \frac{i\bar{z}}{1+|z|^2} u$

Similar computation gives $v \longmapsto dv + \frac{\bar{w}}{1+|w|^2} v\,dw$

(iv) re-visiting ☹ in this example

$$u \longmapsto du + \frac{\bar z}{1+|z|^2} u\, dz$$

transform $\downarrow$ $\qquad\qquad$ $\downarrow$ transform

$$v = uz \qquad \left(du + \frac{\bar z}{1+|z|^2} u\, dz\right) z = (du)\, z + \frac{|z|^2}{1+|z|^2} u\, dz$$

$$= d(vw)\, w^{-1} + \frac{1}{1+|w|^2} vw\, d\left(\frac{1}{w}\right) \qquad d\left(\frac{1}{w}\right) = -\frac{dw}{w^2}$$

$$= dv + \frac{dw}{w} v - \frac{dw}{w} \frac{1}{1+|w|^2} v$$

$$= dv - \frac{\bar w}{1+|w|^2} v\, dw$$

Use $u = vw$, $z = w^{-1}$

of course consistent

<u>the same lesson</u> need zeroth order term to absorb the ambiguity

§ III. formal definition: connection / covariant derivative

$\mathfrak{X}(M)$ = { smooth vector field } = { smooth sections of TM }

$\Gamma(E)$ = { smooth sections of E }

Note that multiplication by a smooth function, $f \in \mathcal{C}^\infty(M)$, is still a section (of the vector bundle)

<u>defn</u> A connection on E is a bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E) \qquad \text{which satisfies}$$
$$(X, s) \longmapsto \nabla_X s$$

(i) $\nabla_{f\cdot X} s = f \cdot (\nabla_X s)$ $\quad$ X is the direction, and is NOT differentiated

(ii) $\nabla_X (f \cdot s) = X(f) \cdot s + f \cdot \nabla_X s$ $\quad$ Leibniz rule

$\forall f \in \mathcal{C}^\infty(M)$, $X \in \mathfrak{X}(M)$, $s \in \Gamma(E)$

$\nabla_X s$ is called the covariant derivative of s in X.

<u>lem</u> connection always exists ($\forall$ vector bundle E)

pf: $\{\mathcal{U}_\alpha\}$: open cover of M. with $\varphi_\alpha : E|_{\mathcal{U}_\alpha} \cong \mathcal{U}_\alpha \times \mathbb{R}^k$

Let $\{\rho_j\}_{j=1}^{\infty}$ be the partition of unity subordinate to $\{\mathcal{U}_\alpha\}_\alpha$

For each j, choose $\alpha(j)$ such that $\mathcal{U}_{\alpha(j)} \supset \operatorname{supp} \rho_j$

For local sections defined on $\mathcal{U}_\alpha$, $s : \mathcal{U}_\alpha \to E \xrightarrow{\varphi_\alpha} \mathcal{U}_\alpha \times \mathbb{R}^k \to \mathbb{R}^k$

Then, $(X, s) \longmapsto \varphi_\alpha^{-1}(X(\varphi_\alpha \circ s))$ (abuse notation) $\varphi_\alpha \circ s$

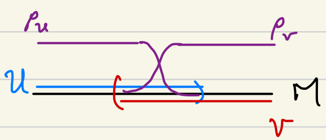
is a connection for $E|_{\mathcal{U}_\alpha}$. Denote it by ∇^α

Now, consider $\nabla : (X, s) \longmapsto \sum_j \rho_j \nabla_X^{\alpha(j)} s$ restrict them on $\mathcal{U}_{\alpha(j)}$

Since $\sum_j \rho_j \cdot f|_{\mathcal{U}_{\alpha_j}} = f$ and $\sum_j \rho_j \cdot d(f|_{\mathcal{U}_{\alpha_j}}) = df$,

one can check that ∇ defines a connection. #

<u>discussion</u> $M = \mathcal{U} \cup \mathcal{V}$, E : given by $g_{uv} : \mathcal{U} \cap \mathcal{V} \to GL(k; \mathbb{R})$



$s|_{\mathcal{U}}$ is given by $s(y) : \mathcal{U} \to \mathbb{R}^k$

Focus on $\mathcal{U}$. use $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^k$

- contribution from $\mathcal{U}$: $\rho_u \, X(s)$
- contribution from $\mathcal{V}$ (on $\mathcal{U} \cap \mathcal{V}$) in terms of $E|_{\mathcal{V}} \cong \mathcal{V} \times \mathbb{R}^k$

the section is $g_{vu}(y) \cdot s : \mathcal{U} \cap \mathcal{V} \to \mathbb{R}^k$

derivative $\Rightarrow$ $X(g_{vu} \cdot s) = X(g_{vu}) \cdot s + g_{vu} \cdot X(s)$

express it by φ_u $\Rightarrow$ $g_{uv} \cdot (X(g_{vu} \cdot s)) = g_{uv} X(g_{vu}) \cdot s + X(s)$

$\Rightarrow$ $\rho_v \cdot X(s) + \rho_v \cdot g_{uv} \cdot X(g_{uv}^{-1}) \cdot s$

Sum $=$ $X(s) + \underbrace{\rho_v \, g_{uv} \, X(g_{uv}^{-1})}_{k \times k \text{ matrix}} \, s$

taking directional derivative in X

$k \times 1$ vector valued function

<u>non-uniqueness</u> It turns out there are many different connections.

§ IV. function-linear is tensorial

$\{E_\mu\}_{\mu=1}^m$, F : vector bundles over M.

Let $A \in \Gamma(\mathrm{Hom}(E_1 \otimes \cdots \otimes E_m, F))$.

Namely, $A_p : E_1|_p \times \cdots \times E_m|_p \longrightarrow F|_p$ is multi-linear.

and vary smoothly in p — ∞-dimensional vector space

$\rightsquigarrow$ It induces $\mathcal{A} : \Gamma(E_1) \times \cdots \times \Gamma(E_m) \longrightarrow \Gamma(F)$

$$(s_1, \cdots\cdots, s_m) \longmapsto A_p(s_1(p), \cdots, s_m(p))$$

Note that A is not only $\mathbb{R}$ (or $\mathbb{C}$)-multi-linear, but also function-multi-linear:

$$\mathcal{A}(s_1, \cdots, f s_\nu, \cdots, s_m) = f\, \mathcal{A}(s_1, \cdots, s_m)$$

$$\forall\, \nu \in \{1, 2, \cdots, m\}, \quad s_\mu \in \Gamma(E_\mu), \quad f \in C^\infty(M)$$

<u>lemma</u> This construction gives all function-multi-linear map from $\Gamma(E_1) \times \cdots \times \Gamma(E_m)$ to $\Gamma(F)$

<u>rmk</u> $M =$ open ball in $\mathbb{R}^n$, $E = F =$ trivial $\mathbb{R}^1$-bundle

$\Gamma(E) = \Gamma(F) = C^\infty(M)$

$\frac{\partial}{\partial x^i} : \Gamma(E) \longrightarrow \Gamma(F)$ is $\mathbb{R}$-linear, but NOT function-linear

constant functions

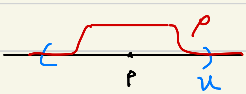
pf: Suppose that $\mathcal{A} : \Gamma(E_1) \times \cdots \times \Gamma(E_m) \to \Gamma(F)$ is $C^\infty(M)$-multi-linear

The key step is to show that $\mathcal{A}(s_1, \cdots, s_m)|_p$ only depends on $(s_1(p), \cdots, s_m(p))$

Choose a coordinate neighborhood $\mathcal{U}$ at p, such that $p \leftrightarrow x = 0$

and $E_\mu|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^{k_\mu}$, $F|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^\ell$

Choose $\rho \in C^\infty(M)$, such that $\begin{cases} \mathrm{supp}(\rho) \subset \mathcal{U}. \\ \rho = 1 \text{ on a neighborhood of } p \end{cases}$



Note that $\mathcal{A}(s_1, \cdots, s_\nu, \cdots, s_m)|_p = \mathcal{A}(s_1, \cdots, \rho s_\nu, \cdots, s_m)|_p$

<u>claim</u> If $(s_1(p), \cdots, s_m(p)) = (\tilde{s}_1(p), \cdots, \tilde{s}_m(p))$,

then $\mathcal{A}(s_1, \cdots, s_m)|_p = \mathcal{A}(\tilde{s}_1, \cdots, \tilde{s}_m)_p$

It suffices to show that for any $\nu \in \{1, \cdots, m\}$

$$\mathfrak{A}(s_1, \cdots, s_\nu, \cdots, s_m)|_p = \mathfrak{A}(s_1, \cdots, \tilde{s}_\nu, \cdots, s_m)|_p \quad \text{if} \quad s_\nu(p) = \tilde{s}_\nu(p)$$

$$\Leftrightarrow \quad \mathfrak{A}(s_1, \cdots, \sigma_\nu, \cdots, s_m)|_p = 0 \quad \text{if} \quad \sigma_\nu(p) = 0$$

By Taylor. $\rho\sigma_\nu = x^j h_j(x)$, $\rho^3 \sigma_\nu = \underbrace{(\rho x^j)}_{C^\infty(M)} \cdot \underbrace{(\rho h_j(x))}_{\Gamma(E)}$

$$\begin{aligned} &\mathfrak{A}(s_1, \cdots, \sigma_\nu, \cdots, s_m)|_p \\ &= \mathfrak{A}(s_1, \cdots, \rho^3\sigma_\nu, \cdots, s_m)|_p = \cancelto{0}{(\rho x^j)}|_p \cdot \mathfrak{A}(s_1, \cdots, \rho h_j, \cdots, s_m)|_p = 0 \end{aligned}$$

※

<u>discussion</u> • connection is NOT function-linear

- Traditionally, a tensor of type (p, q) is a smooth section of $(\otimes^p TM) \otimes (\otimes^q T^*M) \cong ((\otimes^p T^*M) \otimes (\otimes^q TM))^*$
$$\cong \mathrm{Hom}((\otimes^p T^*M) \otimes (\otimes^q TM); \mathbb{R})$$

$\Leftrightarrow$ function-multi-linear functional on

$$\underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{p} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{q}$$

it eats p 1-forms and q vector fields.

<u>prop</u> $E \to M$ vector bundle. the space of all connections is an <u>affine space</u> modelled on $\Gamma(T^*M \otimes \mathrm{End}(M))$

vector space without origin

pf: Suppose that $\tilde{\nabla}$ and ∇ are two connection on E

Then, $\tilde{\nabla} - \nabla : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$ is $C^\infty(M)$-bilinear

$$\begin{aligned} \tilde{\nabla}_X(fs) - \nabla_X(fs) &= (X(f) \cdot s + f \tilde{\nabla}_X s) - (X(f) \cdot s + f \nabla_X s) \\ &= f \cdot (\tilde{\nabla}_X s - \nabla_X s) \end{aligned}$$

According to the lemma, $\tilde{\nabla} - \nabla$ is given by some smooth section of $\mathrm{Hom}(TM \otimes E; E) \cong T^*M \otimes \mathrm{End}(E)$

It is easy to check that $\nabla + \Gamma(T^*M) \otimes \mathrm{End}(M))$ always define a connection ※

§V. how to do computation

$E \to M$, with a connection ∇.

$\mathcal{U}$: open set with $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^k$. What does ∇ look like over $\mathcal{U}$?

$\Gamma(E|_{\mathcal{U}}) \cong \{ \mathbb{R}^k\text{-valued functions} \}$

d : component wise exterior derivative defines a connection

$\Rightarrow \nabla = d +$ a section of $T^*\mathcal{U} \otimes \underbrace{\mathrm{End}(E|_{\mathcal{U}})}_{\cong\ k\times k\text{-matrices}}$

$\Rightarrow \nabla = d + \underbrace{\sum_j A_j(x)\, dx^j}_{\text{denote it by } A_{\mathcal{U}}}$ each $A_j(x)$ is a $k \times k$ matrix

transition? $E|_V \cong V \times \mathbb{R}^k$, $\nabla = d + A_V$

$\mathbb{R}^k$-valued function $\rightsquigarrow$

$$\begin{tikzcd}
\mathcal{U} & & V \\[-20pt]
s \arrow[rr, mapsto] \arrow[d, "\nabla"'] & & g\cdot s \arrow[d, "\nabla"] \\
ds + A_{\mathcal{U}}\cdot s \arrow[rr, mapsto] & & d(gs) + A_V \cdot g \cdot s
\end{tikzcd}
\qquad g = g_{V\mathcal{U}}$$

$\Rightarrow g\cdot ds + g A_{\mathcal{U}} \cdot s = d(gs) + A_V \cdot g \cdot s \quad \forall s$

$= (dg)\cdot s + g \cdot ds + A_V \cdot g \cdot s$

$\Rightarrow g A_{\mathcal{U}} = dg + A_V \cdot g$

$\Rightarrow A_{\mathcal{U}} = g^{-1} dg + g^{-1} A_V g$: this is how the local expression of the connection transforms