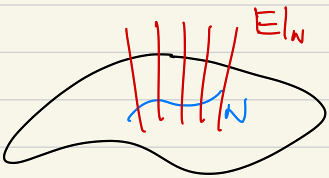
§I. pull-back bundle

0) $N \subset M$ submanifold, $E \to M$ vector bundle

We may "restrict" E on N



M

Via defn 2:

$\begin{cases} \{\mathcal{U}_\alpha\}: \text{open cover of } M \\ g_{\alpha\beta}: \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to Gl(k; \mathbb{R}) \end{cases}$ transitions

$\rightsquigarrow \begin{cases} \mathcal{U}_\alpha' = \mathcal{U}_\alpha \cap N : \text{open cover of } N \\ g_{\alpha\beta}' = g_{\alpha\beta}|_{\mathcal{U}_\alpha' \cap \mathcal{U}_\beta'} \end{cases}$

Via defn 1: $E|_N = \pi^{-1}(N)$ It suffices to show that $E|_N$ is a smooth manifold.

recall $\forall p \in N^n \subset M^m$, $\exists\, \mathcal{U}$: coordinate nbd of $\mathcal{U}$ in M $(x^1, \cdots, x^m)$

such that $\mathcal{U} \cap N = \{x^{n+1} = x^{n+2} = \cdots = x^m = 0\}$

In fact, N needs not be a submanifold

1) <u>input</u> $E \xrightarrow{\pi} M$: a vector bundle, $N \xrightarrow{f} M$: a smooth map (no condition on their dimensions)

<u>output</u> $f^*E \to N$ the pull-back bundle

Its rank is the same as that of E

$$\begin{tikzcd} & \mathbb{R}^k \arrow[d] & \\ & E \ni e \arrow[d, "\pi"] \\ N^n \ni p \arrow[r, "f"] & M^m \end{tikzcd}$$

Note that $\forall p \in N$, $f^*E|_p$ "=" $E_{f(p)}$

Let $f^*E = \{(p, e) \in N \times E \mid f(p) = \pi(e)\}$

$E \to M \rightsquigarrow$ naturally regard it as $E \to N \times M$

restrict it on $\Gamma_f = \{(p, f(p)) \mid p \in N\}$

(embedding $N \hookrightarrow N \times M$)

$\boxed{\text{DIY}}$ describe the pull-back bundle via defn 2

example $f: N \to M$ constant map
$p \mapsto q_0$ $\Rightarrow f^*E = N \times \mathbb{R}^k$

§II. subbundle and quotient bundle

subbundle of E intuitively means that there is a vector subspace (of the same dimension) of each E_p

defn $E \xrightarrow{\pi} M$, a subbundle of E is a vector F with

$$\begin{tikzcd} F \arrow[rr, hook, "\iota"] \arrow[dr, "\pi"'] & & E \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

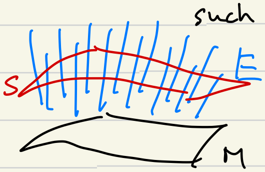
so that
- diagram commutes
- ι : embedding
- $F_p \xrightarrow{\iota_p} E_p$ linear

ι_p : injective

$\boxed{\text{question}}$ We can form E_p / F_p $\forall p \in M$. Will it constitute a vector bundle?

We are going to construct it via defn 2.

defn a (smooth) section of E is a smooth map $s: M \to E$ such that $\pi \circ s = 1_M$



Namely. $\forall p \in M$. there is a vector $s(p)$ in E_p. and they vary smoothly in p

lem a local trivialization $E|_U \cong U \times \mathbb{R}^k$ is equivalent to k locally defined smooth sections, which form a basis for each E_p

$(p, e_k) = s_k$

pf: by definition ⋯ #

call them local trivializing sections

rmk
- a vector bundle is trivial if and only if it admits global trivializing sections
- $M \hookrightarrow E$ naturally by the "zero section".

Back to $F \overset{\iota}{\subset} E \rightsquigarrow$ define E/F

i) $\forall p \in M \ \exists \mathcal{U}$: neighborhood of p $\quad \to \begin{cases} F|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^k \\ E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^\ell \end{cases}$

$$\begin{array}{ccccc} \mathcal{U} \times \mathbb{R}^k \cong F|_{\mathcal{U}} & \xrightarrow{\iota} & E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^\ell \\ (q, e_j) \quad s_j & \longmapsto & \end{array}$$

$\Rightarrow \{(\iota \circ s_j)\}$ is linearly independent in E_q $\forall q \in \mathcal{U}$

ii) By linear algebra, $\exists \{w_{k+1}, \cdots, w_\ell\} \in E_p$ such that $\{\iota \circ s_1(p), \cdots, \iota \circ s_k(p), w_{k+1}, \cdots, w_\ell\}$ form a basis of E_p

Take a smooth extension of $\{w_{k+1}, \cdots, w_\ell\}$ over $\mathcal{U}$.

Denote it by $\{\tilde{s}_{k+1}, \cdots, \tilde{s}_\ell\}$ For instance, extend them "constantly" by $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^\ell$

iii) By looking at $\mathbb{R}^\ell \xrightarrow{\det} \mathbb{R}^1$,
$\{\iota \circ s_1, \cdots, \iota \circ s_k, \tilde{s}_{k+1}, \cdots, \tilde{s}_\ell\}$ form local trivialing sections of E over some neighborhood of p (shrink $\mathcal{U}$)

iv) Hence, $\exists$ open cover $\{\mathcal{U}_\alpha\}$, and $\varphi_\alpha : E|_{\mathcal{U}_\alpha} \cong \mathcal{U}_\alpha \times \mathbb{R}^\ell$ such that $\varphi_\alpha : F|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times (\mathbb{R}^k \times \{0\})$

It follows that
$$g^E_{\alpha\beta} = \begin{array}{c} \\ k \\ \ell-k \\ \\ \end{array}\!\!\begin{array}{c} \begin{bmatrix} g^F_{\alpha\beta} & g''_{\alpha\beta} \\ 0 & g'_{\alpha\beta} \end{bmatrix} \\ \begin{array}{cc} k & \quad \ell-k \end{array} \end{array}.$$

<u>claim</u> $\{g'_{\alpha\beta}\}$ satisfies the cocycle condition

$$g^E_{\alpha\beta} \cdot g^E_{\beta\gamma} = g^E_{\alpha\gamma} \Rightarrow \begin{bmatrix} g^F_{\alpha\beta} & g''_{\alpha\beta} \\ 0 & g'_{\alpha\beta} \end{bmatrix} \begin{bmatrix} g^F_{\beta\gamma} & g''_{\beta\gamma} \\ 0 & g'_{\beta\gamma} \end{bmatrix} = \begin{bmatrix} g^F_{\alpha\gamma} & g''_{\alpha\gamma} \\ 0 & g'_{\alpha\gamma} \end{bmatrix}$$

<u>example</u> / <u>recap</u> $\quad f : N \to M$ smooth map (no further assumption)

$f_*|_p : T_pN \to T_{f(p)}M$

$$\begin{tikzcd} TN \arrow[rr, "f_*"] \arrow[dr] & & f^*TM \arrow[dl] \\ & N & \end{tikzcd}$$
$\rightsquigarrow$ morphism of bundles over M

If $f_*|_p$ is injective $\forall p$ ($\Leftrightarrow f$ is an immersion)

TN can be regarded as a subbundle of $f^* TM$

In this case, the quotient bundle is called the normal bundle of $f(N)$ in M.

<u>rmk</u> Later on, we will introduce the inner product for vector bundles, and we will see that the quotient construction is the same as "normal" in the usual sense.

§ III algebraic constructions.

(pseudo-) <u>thm</u> For vector bundles over M, can perform linear algebraic construction $\forall p \in M$ to get another vector bundle

It is easier to do it via defn 2. Here are some examples

1) <u>dual bundle</u>

$$V_1 \xrightarrow{A} V_2 \quad \Rightarrow$$

$$\rightsquigarrow \quad V_1^* \xleftarrow{A^T} V_2^*$$

transpose in terms of dual basis

Over $\mathcal{U} \cap \mathcal{V}$

$$E|_{\mathcal{V}} \cong \mathcal{V} \times \mathbb{R}^k \longrightarrow \mathcal{U} \times \mathbb{R}^k \cong E|_{\mathcal{U}}$$

$$(x, v) \longmapsto (x, g_{uv}(x) v)$$

$$\rightsquigarrow E^*|_{\mathcal{V}} \cong \mathcal{V} \times \mathbb{R}^k \longleftarrow \mathcal{U} \times \mathbb{R}^k \cong E^*|_{\mathcal{U}}$$

$$(x, (g_{uv}(x))^T w) \longleftarrow\!\shortmid (x, w)$$

Define E^* by the transitions $g^*_{uv} = (g_{uv}^T)^{-1}$, which still satisfies the cocycle condition

2) <u>tensor product</u> recall V, W : vector space

① $V \otimes W$ abstractly : $\exists\ \varphi : V \times W \to V \otimes W$ bilinear such that any bilinear map $F : V \times W \to Z$ factors through φ. Namely

$$\begin{tikzcd} V \times W \arrow[r, "\varphi"] \arrow[dr, "F"'] & V \otimes W \arrow[d, "f"] \\ & Z \end{tikzcd}$$

(ii) explicitly. $\{v_i\}$: basis for V. $\{w_j\}$ basis for W

$V \otimes W$ has basis $\{v_i \otimes w_j\}$. $\dim V \otimes W = \dim V \cdot \dim W$

(iii) $V_1 \xrightarrow{A} V_2$, $W_1 \xrightarrow{B} W_2$ linear maps

$\rightsquigarrow V_1 \otimes W_1 \xrightarrow{A \otimes B} V_2 \otimes W_2$

$A v_i = \sum_k a_i^k \tilde{v}_k$, $B w_j = \sum_\ell b_j^\ell \tilde{w}_\ell$ $\Rightarrow (A \otimes B)(v_i \otimes w_j)$

$\tilde{v}_k$: basis for V_2 $\tilde{w}_\ell$: basis for W_2 $= \sum_{k,\ell} a_i^k b_j^\ell (\tilde{v}_k \otimes \tilde{w}_\ell)$

Ordering the basis suitably. $A \otimes B = \begin{bmatrix} \boxed{a_1^1 B} & \cdots & \boxed{a_1^m B} \\ & & \\ \boxed{a_m^1 B} & \cdots & \boxed{a_m^m B} \end{bmatrix}$

Say, $\dim V_1 = m = \dim V_2$

each $\boxed{}$ has the same size as B.

(iv) E, F : vector bundles over M

Define $E \otimes F$ by the transitions $g_{uv}^{E \otimes F} = \underset{A}{g_{uv}^E} \otimes \underset{B}{g_{uv}^F}$

$\boxed{\text{check}}$ Since g_{uv}^E, g_{uv}^F satisfy the cocycle condition.

so does $g_{uv}^{E \otimes F}$

3) others.

(0) E, F : vector bundles over M

$E \oplus F$ is defined by the transitions $g_{uv}^{E \oplus F} = \begin{bmatrix} g_{uv}^E & 0 \\ 0 & g_{uv}^F \end{bmatrix}$

(i) $\operatorname{rank} E = k \rightsquigarrow \Lambda^\ell E$: rank $= \binom{k}{\ell}$

$\Lambda^0 E = M \times \mathbb{R}$, $\Lambda^1 E = E$

$\Lambda^k E$: rank $= 1$, $\boxed{\text{check}}$ $g_{uv}^{\Lambda^k E} = \det(g_{uv}^E)$

Note that taking det preserves the cocycle condition

(ii) $\operatorname{Hom}(E, F) = F \otimes E^*$ (input E, output F)

(iii) <u>recap</u> $TM \rightsquigarrow T^*M = (TM)^*$

$\Lambda^\ell M = \Lambda^\ell(T^*M)$. its sections are differential ℓ-forms on M