§I. some motivations

1° recall Poincaré – Hopf (simplest case)

Σ: closed, oriented surface.

V: tangent vector field on Σ with isolated zeros

Then, $\displaystyle\sum_{p \in V^{-1}(0)} i(V;p) = 2 - 2g(\Sigma)$

sum (local indices) = topological quantity

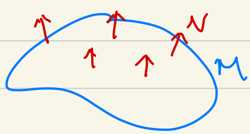
$\rightsquigarrow$ tangent vector fields reflects the topology of the manifold in some way

2° X: manifold $\supset$ M : submanifold

The first step to understand the inclusion is to look at the "linearization" of its tubular neighborhood

e.g. $M^n \subset \mathbb{R}^{n+1}$. (closed and oriented)

$\Leftrightarrow \exists N : M \to \mathbb{R}^{n+1}$ unit normal vector



$$M \times (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^{n+1}$$
$$(p, t) \longmapsto p + t N(p) \quad \text{is an embedding}$$

Namely, an open neighborhood of M is diffeomorphic to $M \times \mathbb{R}^1$.

But in general this is NOT the case

(You may recall the last two questions in the final)

§II. the definitions

Basically, there is a base manifold M, and each $p \in M$ carries a vector space. These vector spaces "vary smoothly" in p.

<u>defn 1</u> M: smooth manifold, a rank k vector bundle (over $\mathbb{R}$ or $\mathbb{C}$) is a smooth manifold E with a smooth map $\pi : E \to M$ such that

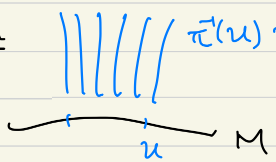
i) $\forall p \in M$, $E_p = \pi^{-1}(p)$ has a structure of k-dim'l vector space

ii) local triviality $\forall p \in M$, $\exists$ open neighborhood $\mathcal{U}$ of p in M and a diffeomorphism

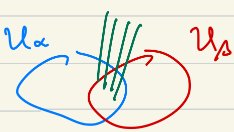
$$\varphi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \subset E \longrightarrow \mathcal{U} \times \mathbb{R}^k \ (\text{or } \mathbb{C}^k)$$

so that $\varphi_{\mathcal{U}}(E_q) = \{q\} \times \mathbb{R}^k$ $\forall q \in \mathcal{U}$ and it is a linear isomorphism

<u>discussion</u>



We can obtain an open cover $\{\mathcal{U}_\alpha\}$ of M, together with $\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \xrightarrow{\sim} \mathcal{U}_\alpha \times \mathbb{R}^k$



For $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$

$$\begin{tikzcd} E_x \arrow[r, bend left, "\varphi_\alpha"] \arrow[r, bend right, "\varphi_\beta"'] & \{x\} \times \mathbb{R}^k \end{tikzcd}$$

$\varphi_\alpha, \varphi_\beta$: two linear isomorphisms between E_x and $\{x\} \times \mathbb{R}^k$

$\Rightarrow \varphi_\alpha \circ \varphi_\beta^{-1} : \{x\} \times \mathbb{R}^k \circlearrowleft$ linear isomorphism

$$\pi^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

$$\Rightarrow \quad \varphi_\alpha \circ \varphi_\beta^{-1} : (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{R}^k \longrightarrow (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{R}^k$$

$$(x, v) \longmapsto (x, g_{\alpha\beta}(x) v)$$

where $g_{\alpha\beta}$ smooth map from $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ to $Gl(k;\mathbb{R})$

$\Rightarrow$ vector bundle structure gives the data : $g_{\alpha\beta} \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta, Gl(k;\mathbb{R}))$

- Naturally, $g_{\alpha\alpha} \equiv 1 \quad \forall \alpha$ for $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$
- $g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \quad \forall x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$ open subset of $\mathbb{R}^{k^2}$ and a group

LHS : given by $(\varphi_\alpha \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\gamma^{-1})$

RHS : given by $(\varphi_\alpha \circ \varphi_\gamma^{-1})$ on $\{x\} \times \mathbb{R}^k$

This is called the cocycle condition.

defn 2 A vector bundle over M consists of the data: open cover $\{\mathcal{U}_\alpha\}$ and $g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to Gl(k;\mathbb{R})$ smooth

lemma two definitions are equivalent

pf: $1 \Rightarrow 2$ as in the discussion

$2 \Rightarrow 1$ Define E by

$$\coprod_\alpha \mathcal{U}_\alpha \times \mathbb{R}^k \Big/ (x, v) \sim (x, g_{\alpha\beta}(x) v) \quad \forall x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$$

$(x,v) \in \mathcal{U}_\beta \times \mathbb{R}^k$, $(x, g_{\alpha\beta}(x)v) \in \mathcal{U}_\alpha \times \mathbb{R}^k$

The cocycle condition $\Rightarrow$ $\sim$ is an equivalent relation

$\mathcal{U}_\alpha \times \mathbb{R}^k \longrightarrow \mathcal{U}_\alpha \subset M$ descends to $\pi : E \to M$

... check details by yourself ... ※

terminologies E : total space π : projection

M : base space $\pi^{-1}(p) = E_p$: fiber at p

$\varphi_\mathcal{U} : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{R}^k$: (local) trivialization

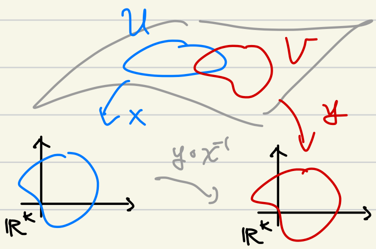
§ III. basic examples / constructions

from definitions, two ways $\overset{1}{\rightsquigarrow}$ construct E and π

$\overset{}{\underset{2}{\rightsquigarrow}}$ construct $g_{\alpha\beta}$ (and $\{U_\alpha\}$)

↙ warm-up / review

1a) tangent bundle : choose a coordinate cover of M $\{(\mathcal{U}, x)\}$ $\quad x : \mathcal{U} \to \mathbb{R}^n$ homeomorphic to an open subset



$y \circ x^{-1}$ is smooth on where it is defined

tangent vector at $q \in \mathcal{U} \cap V$ in terms of x-coordinate

$$\sum_j u^j \frac{\partial}{\partial x^j} = \sum_{j,k} u^j \frac{\partial y^k}{\partial x^j} \frac{\partial}{\partial y^k} = \sum_k v^k \frac{\partial}{\partial y^k}$$

$$\Rightarrow \ (x, \{u^j\}) \sim \left(y, \left\{v^k = \frac{\partial y^k}{\partial x^j} u^j\right\}\right)$$

Namely,

$$g_{vu}(x) = \begin{bmatrix} \partial y^1/\partial x^1 & \cdots & \partial y^1/\partial x^n \\ \vdots & \ddots & \vdots \\ \partial y^n/\partial x^1 & \cdots & \partial y^n/\partial x^n \end{bmatrix}$$

$\boxed{\text{check}}$ cocycle condition $\Longleftrightarrow$ chain rule

1b) cotangent bundle : $\displaystyle\sum_j \alpha_j \, dx^j = \sum_{j,k} \alpha_j \frac{\partial x^j}{\partial y^k} dy^k = \sum_k \beta_k \, dy^k$

$$\Rightarrow \ (x, \{\alpha_j\}) \sim \left(y, \left\{\beta^k = \frac{\partial x^j}{\partial y^k} \alpha_j\right\}\right)$$

Namely,

$$g^*_{vu}(x) = \begin{bmatrix} \partial x^1/\partial y^1 & \cdots & \partial x^n/\partial y^1 \\ \vdots & & \vdots \\ \partial x^1/\partial y^n & & \partial x^n/\partial y^n \end{bmatrix} \quad \text{at } y = y(x)$$

Compare 1a) and 1b): from the chain rule on
$(x \circ y^{-1}) \circ (y \circ x^{-1})$ = the identity map on $x \in \mathbb{R}^n$

$$\underbrace{\begin{bmatrix} \partial x^1/\partial y^1 & \cdots & \partial x^1/\partial y^n \\ \vdots & \ddots & \vdots \\ \partial x^n/\partial y^1 & \cdots & \partial x^n/\partial y^n \end{bmatrix}}_{= (g^*_{vu}(x))^T} \cdot \underbrace{\begin{bmatrix} \partial y^1/\partial x^1 & \cdots & \partial y^1/\partial x^n \\ \vdots & \ddots & \vdots \\ \partial y^n/\partial x^1 & \cdots & \partial y^n/\partial x^n \end{bmatrix}}_{= g_{vu}(x)} = 1$$

$\Rightarrow$ $g^*_{vu} = ((g_{vu})^T)^{-1}$ still obeys the cocycle condition
of course, one can also check it directly by the chain rule
But we will see this construction for general E later.

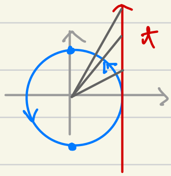
2) tautological bundle over $\mathbb{RP}^n$

$$\mathbb{RP}^n = \{\text{all lines in } \mathbb{R}^{n+1}\} = \mathbb{R}^{n+1} \setminus \{0\} \big/ \ell \sim s\ell \quad s \neq 0$$

$$E \subset \mathbb{RP}^n \times \mathbb{R}^{n+1}$$
$$\rotatebox{90}{$\cong$}\ \{([\ell], v) \mid v \text{ is parallel to } \ell\}$$

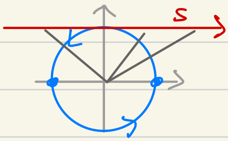
$\boxed{\text{claim}}$ E is an rank 1 real vector bundle over $\mathbb{RP}^n$

e.g. $n=1$, $\mathbb{RP}^1 \cong S^1$



$$\begin{array}{ccc} \mathbb{R}^1 & \longrightarrow & \mathcal{U} = \mathbb{RP}^1 \setminus \{y\text{-axis}\} \\ t & & [1, t] \end{array} \qquad E|_{\mathcal{U}} = u\,(1, t)$$

$$\rightsquigarrow \quad \pi^{-1}(\mathcal{U}) \xleftarrow{\varphi^{-1}} \mathcal{U} \times \mathbb{R}^1$$
$$\left([1, t],\ u(1,t)\right) \longleftarrow ([1,t], u)$$



$\mathbb{R}^1 \longrightarrow \mathcal{V} = \mathbb{RP}^1 \setminus \{x\text{-axis}\}$

$s \longmapsto [s, 1]$

$E|_{\mathcal{V}} = v(s, 1)$

$[1, t] = [\frac{1}{t}, 1]$

$\mathcal{U} \cap \mathcal{V} \cong$ two open intervals (on half line)

$\mathbb{R}^1 \setminus \{0\} \xrightarrow{\sim}$ $\qquad \xleftarrow{\sim} \mathbb{R}^1 \setminus \{0\}$

$t \longmapsto s = 1/t$

$g_{vu} = ?$ $\quad ([1, t], u(1, t)) = ([s, 1], v(s, 1))$

$= ([\frac{1}{t}, 1], v(\frac{1}{t}, 1))$

$\Rightarrow v = t u \qquad g_{vu} = t$

$\boxed{\text{think}}$ $(n = 1)$, $E \cong$ Möbius band

$\boxed{\text{question}}$ How to show $E \not\cong S^1 \times \mathbb{R}^1$?

In this case, one can use intermediate value theorem

In general, it is easier to study it for C^k-bundles.

$\rightsquigarrow$ the theory of characteristic class (Chern class)

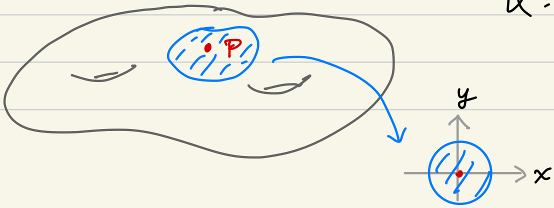
will also do $\mathbb{R}^k$-bundles

Here are two special constructions, which will serve as fundamental examples in the study of first and second Chern classes, respectively.

3) Σ : (closed) oriented surface. Choose $p \in \Sigma$

$\mathcal{U}$: a coordinate neighborhood at p such that $dx \wedge dy$ is the orientation, and $p \mapsto$ origin

$\mathcal{V} = \Sigma \setminus \{p\} \Rightarrow \Sigma = \mathcal{U} \cup \mathcal{V}$



two open sets $\Rightarrow$ cocycle condition $= \phi$ (only $g_{uv} = g_{vu}^{-1}$)

$\mathcal{U} \cap \mathcal{V} \cong B(0;1) \setminus \{0\} \cong S^1 \times (0,1)$

To construct a $\mathbb{C}^1$-bundle (complex line bundle),

we need to assign a map from $\mathcal{U} \cap \mathcal{V} \to Gl(1;\mathbb{C})$

$\mathcal{U}(1) \subset Gl(1;\mathbb{C}) = \mathbb{C} \setminus \{0\}$ is $\cong S^1 \times (0,\infty)$

Let $g_{uv} : B(0;1) \setminus \{0\} \longrightarrow \mathcal{U}(1)$

$$x+iy = re^{i\theta} \longmapsto e^{i\theta} = \frac{x+iy}{\sqrt{x^2+y^2}}$$

Call the bundle E_1.

$\boxed{\text{question}}$ We can also consider $g_{uv}^n : re^{i\theta} \mapsto e^{in\theta}$

Call the bundle E_n. How do we know E_n are "different" bundles for different n?

4) analogy for $\mathbb{C}^2$-bundles over 4-manifold.

M: (closed) oriented 4-dimensional manifold

We can fix $p \in X$, and choose $\mathcal{V} = M \setminus \{p\}$

$\mathcal{U}$: coordinate neighborhood at p with $p \mapsto 0$ and

$dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$: the orientation.

$\mathcal{U} \cap \mathcal{V} \cong B(0;1) \setminus \{0\} \cong S^3 \times (0,1)$

<u>recall</u> $SU(2)$ is diffeomorphic to S^3

$$= \left\{ \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} \;\middle|\; |z|^2+|w|^2 = 1 \right\}$$

$\rightsquigarrow$ Consider $g_{vu} : \mathcal{U} \cap \mathcal{V} \longrightarrow SU(2) \subset Gl(2;\mathbb{C})$

$$0 \neq (z,w) \longmapsto \frac{1}{\sqrt{|z|^2+|w|^2}} \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix}$$

$\rightsquigarrow$ We can also take its n-th power, and ask the same question.

§IV same bundle?

In 3) of the previous section, we may also take

$$g_{uv} : B(0,1)\setminus\{0\} \longrightarrow GL(1;\mathbb{C}) = \mathbb{C}\setminus\{0\}$$
$$x+iy = re^{i\theta} \longmapsto r^{l}e^{i\theta}$$
$$z \longmapsto |z|^{l-1} z \qquad \text{for } l\in\mathbb{R}$$

Will it give "different" bundles? or "same" bundle?

<u>recall</u> For (smooth) manifold, we always think diffeomorphism manifolds are the same.

We shall only consider smooth maps which respect the vector space structure.

<u>defn</u> E, E' : vector bundle over M. a morphism from E to E' is a smooth map $\psi : E \to E'$ $\ni$ diagram commutes

$$\begin{tikzcd} E \arrow[r, "\psi"] \arrow[dr, "\pi"'] & E' \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

and $\psi|_x : E_x \to E'_x$ is linear

<u>rmk</u> rank $\psi|_x$ (and nullity) needs not be constant

<u>defn</u> ψ is called an isomorphism if $\psi|_x : E_x \to E'_x$ is an isomorphism $\forall x\in M$ (if so, rank E = rank E')

<u>discussion</u> The above notion is based of defn 1 of vector bundle. What does ψ looks like in terms of defn 2?

By taking a common refinement, we may assume $\exists$ open cover $\{U_\alpha\}$ of M

$$\begin{tikzcd} E|_{U_\alpha} \arrow[r, "\varphi_\alpha"] \arrow[d, "\psi"', "\text{morphism}"] & U_\alpha \times \mathbb{R}^k \arrow[d] \\ E'|_{U_\alpha} \arrow[r, "\varphi'_\alpha"] & U_\alpha \times \mathbb{R}^{k'} \end{tikzcd}$$

$\varphi'_\alpha \circ \psi \circ \varphi_\alpha^{-1}(x,v) = (x, T_\alpha(x)v)$

$T_\alpha : U_\alpha \to M(k'\times k, \mathbb{R})$ smooth

It is easier to see the relation between $T_\alpha(x)$ and $T_\beta(x)$:

$$\begin{tikzcd}
(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)\times \mathbb{R}^k \arrow[d] & E|_{\mathcal{U}_\alpha\cap\mathcal{U}_\beta} \arrow[l, "\varphi_\beta"'] \arrow[r, "\varphi_\alpha"] \arrow[d, "\psi"] & (\mathcal{U}_\alpha \cap \mathcal{U}_\beta)\times \mathbb{R}^k \arrow[d] \\
(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)\times \mathbb{R}^{k'} & E'|_{\mathcal{U}_\alpha\cap\mathcal{U}_\beta} \arrow[l, "\varphi_\beta'"'] \arrow[r, "\varphi_\alpha'"] & (\mathcal{U}_\alpha \cap \mathcal{U}_\beta)\times \mathbb{R}^{k'}
\end{tikzcd}$$

$(x, v) \mapsto (x, T_\beta(x) v)$, $\quad (x, g_{\beta\alpha}(x) u) \leftarrow\!\shortmid (x, u)$, $\quad (x, u) \mapsto (x, T_\alpha(x) u)$

$(x, g'_{\beta\alpha}(x)(T_\alpha(x) u)) \leftarrow\!\shortmid (x, T_\alpha(x) u)$

$\Rightarrow \quad T_\beta(x)\, g_{\beta\alpha}(x) = g'_{\beta\alpha}(x)\, T_\alpha(x) \quad \forall x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$

Therefore, $T_\alpha(x) : \mathcal{U}_\alpha \to M(k' \times k ; \mathbb{R})$ satisfying the above relation is equivalent to a morphism.

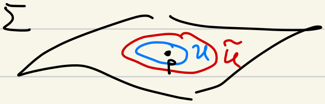
For isomorphism, $k' = k$ and $T_\alpha(x) : \mathcal{U}_\alpha \to Gl(k; \mathbb{R})$

<u>example</u> $\Sigma \ni p$ $\quad \mathcal{U}$ = coordinate neighborhood at p. $\quad \mathcal{V} = \Sigma \setminus \{p\}$

$$g_{uv} : B(0,1)\setminus\{0\} \longrightarrow Gl(1;\mathbb{C}) = \mathbb{C}\setminus\{0\}$$
$$x + iy = re^{i\theta} \longmapsto r^\ell e^{i\theta}$$
$$z \longmapsto |z|^{\ell-1} z \qquad \text{for } \ell \in \mathbb{R}$$

Denote by $\mathbb{C}^1$-bundle by $E_\ell \to \Sigma$. Is E_ℓ isomorphic to E_0?

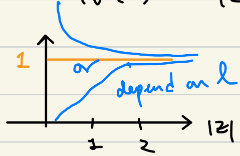
Let us assume $\mathcal{U} \subset \tilde{\mathcal{U}} \subset \Sigma$
$\quad\;\; \mathcal{U} \cong B(0;1)$, $\tilde{\mathcal{U}} \cong B(0;2)$



We need to construct $T_u : \mathcal{U} \to \mathbb{C}^*$, $T_v : \mathcal{V} \to \mathbb{C}^*$

such that $T_u(z)\, |z|^{-1} z = |z|^{\ell-1} z\, T_v(z) \quad \forall z \in \mathcal{U}\cap\mathcal{V}$

Take $T_u(z) \equiv 1$, then $T_v(z) = |z|^{-\ell}$ on $\mathcal{U}\cap\mathcal{V}$



$\exists$ smooth, non zero extension

$\ni T_v \equiv 1$ on $\Sigma \setminus \tilde{\mathcal{U}}$

$\Rightarrow E_\ell$ and E_0 are isomorphic