Jacobi Theta function

$\Sigma_{g \geq 1}$ $\quad \{A_1, \cdots, A_g, B_1, \cdots, B_g\}$: $2g$-loops-cut

$\rightsquigarrow \{\zeta_1, \cdots \zeta_g\}$ basis for the space of holomorphic differentials normalized by $\int_{A_k} \zeta_j = \delta_{jk}$

Denote $[\int_{B_k} \zeta_j]$ by τ_{kj}, $\Rightarrow$ $\underbrace{\tau = \tau^*, \quad \mathrm{Im}\, \tau > 0}_{\tau \,\in\, \text{Siegel upper half space}}$

defn $\quad \theta(z;\tau) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^* \tau n + 2\pi i n^* z)$ (abbreviate as $\theta(z)$ if no confusion)

$\boxed{\text{check}}$ the sum converges exponentially, thus holomorphic on $\mathbb{C}^g$

periodicity? for $m \in \mathbb{Z}^g$

$$\theta(z+m) = \sum_n \exp(\pi i n^* \tau n + 2\pi i n^* z + \overbrace{2\pi i n^* m}^{\exp \nearrow 1}) = \theta(z)$$

$$\theta(z+\tau m) = \sum_n \exp(\pi i n^* \tau n + 2\pi i n^* z + 2\pi i n^* \tau m)$$

$$= \pi i (n+m)^* \tau (n+m) + 2\pi i (n+m)^* z - \pi i m^* \tau m - 2\pi i m^* z$$

$$= \exp(-\pi i m^* \tau m - 2\pi i m^* z)\ \theta(z)$$

Such a function is called quasi-periodic of weight 1

defn a function f is said to be quasi-periodic of weight l if

$$f(z+m) = f(z) \quad \text{and}$$
$$f(z+\tau m) = \exp(l(-\pi i m^* \tau m - 2\pi i m^* z))\ f(z) \qquad \forall m \in \mathbb{Z}^g$$

rmk maximum principle $\Rightarrow$ NO non-trivial holomorphic function on $\mathbb{C}^g$

holomorphic line bundle

$\pi^* H \cong \mathbb{C}^g \times \mathbb{C}$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & H \arrow[d] & \pi^* H \arrow[d] \\ & J(M) & \mathbb{C}^g \arrow[l, "\pi"', "\text{quotient by } J(M)"] \end{tikzcd}$$

these quasi-periodic functions correspond to holomorphic sections of certain H.

$\boxed{\text{exercise}}$ For $p, q \in \mathbb{Q}^g$, let

$$\theta_{p,q}(z;\tau) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i (n+p)^* \tau (n+q) + 2\pi i (n+p)^* (z+q))$$

$$\Rightarrow \begin{cases} \theta_{p,q}(z+m;\tau) = \exp(\underline{2\pi i p^* m})\ \theta_{p,q}(z;\tau) \\ \theta_{p,q}(z+\tau m;\tau) = \exp(\underline{-2\pi i q^* m} - \pi i m^* \tau m - 2\pi i m^* z)\ \theta_{a b}(z;\tau) \end{cases}$$

$\Rightarrow$ quasi-periodic functions of weight l has dimension l^g and have basis

$$\{\theta_{a l^{-1}, 0}(lz, l\tau)\}_a \quad \text{or} \quad \{\theta_{0, b l^{-1}}(z, l^{-1}\tau)\}_b$$

$a = (a_1, \cdots, a_g)$, $a_j \in \{0, 1, \cdots, l-1\}$

same range as a

<u>thm</u> (Riemann) for any $b \in \mathbb{C}^g$, consider the locally single-valued function $f(p) = \theta(b + \varphi(p)) : M \to \mathbb{C}$. Then, either $f \equiv 0$, or

$M \xrightarrow{\varphi} \mathbb{C}^g \xrightarrow{\theta} \mathbb{C}$

$(f) = Q_1 \cdots Q_g$.

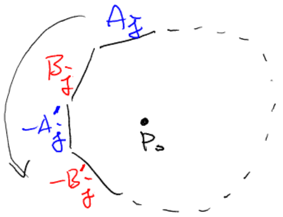
In the second case, $\varphi(Q_1 \cdots Q_g) + b = \Delta \in J(M) = \mathbb{C}^g / L(M)$

Δ: independent of b (Riemann constant vector)

pf: $\varphi : M \to J(M) = \mathbb{C}^g / L(M)$

$p \mapsto \left(\int_{P_0}^{P} \zeta_1, \cdots, \int_{P_0}^{P} \zeta_g \right)$

On the $4g$-gon, φ is well-defined



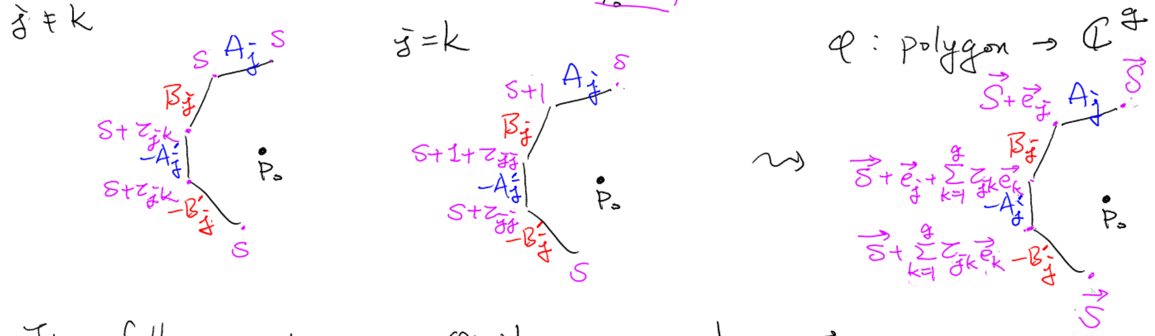
zeros of $f = \theta(z + \varphi(p))$

By argument principle,

$$\#\{\text{zeros within the polygon}\} = \frac{1}{2\pi i} \int_{\partial(\text{polygon})} \frac{df}{f}$$

(counting multiplicity)

We analyzed the value of $\boxed{\int_{P_0}^{P} \zeta_k}$ on $\partial(\text{polygon})$:



It follows that $\varphi(p)|_{B_j'} = \varphi(p)|_{B_j} + \vec{e}_j$

$f(p) = \theta(b + \varphi(p))$

$$\Rightarrow f(p)|_{B_j'} = f(p)|_{B_j} \quad \Rightarrow \quad \frac{df}{f}\Big|_{B_j'} = \frac{df}{f}\Big|_{B_j}$$

$$\varphi(p)|_{A_j'} = \varphi(p)|_{A_j} + \tau e_j$$

j-component of φ

$$\Rightarrow f(p)|_{A_j'} = \exp\Big(\underbrace{-\pi i\, e_j^{*} \tau e_j - 2\pi i\, e_j^{*}(b}_{\text{constant}} + \varphi(p))\Big) f(p)|_{A_j}$$

$$\Rightarrow \frac{df}{f}\Big|_{A_j'} = \frac{df}{f}\Big|_{A_j} - 2\pi i\, \zeta_j$$

Therefore, $$\frac{1}{2\pi i} \int_{\partial(\text{polygon})} \frac{df}{f} = \frac{1}{2\pi i} \sum_{j=1}^{g} \int_{A_j + B_j - A_j' - B_j'} \frac{df}{f} = \sum_{j=1}^{g} \int_{A_j} \zeta_j = g$$

$$\Rightarrow f(p) = \theta(b + \varphi(p)) \text{ has } g \text{ zeros}$$

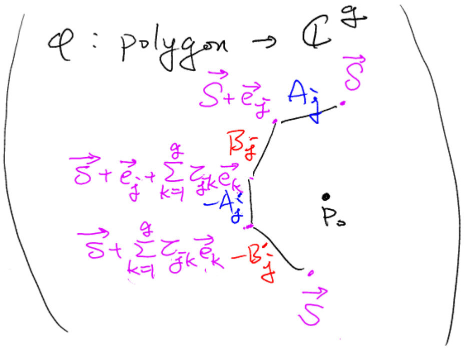
To study the image $Q_1, \cdots, Q_g$ (zeros of $f(P)$) under φ,

$\varphi = \left(\int_{P_0}^{P} \zeta_1, \cdots, \int_{P_0}^{P} \zeta_g \right)$: column vector, g-tuple of functions

we can use $$\frac{1}{2\pi i} \int_{\partial(\text{polygon})} \varphi \frac{df}{f} = \sum_{j=1}^{g} \varphi(Q_j)$$

$$\frac{1}{2\pi i}\int_{\partial(\text{polygon})} \varphi \frac{df}{f} = \frac{1}{2\pi i}\sum_{j=1}^{g}\left[\int_{A_j}\left(\varphi\frac{df}{f} - (\varphi+\tau e_j)\left(\frac{df}{f} - 2\pi i\, \zeta_j\right)\right) + \int_{B_j}\left((\varphi^- + e_j)\frac{df^-}{f^-} - \varphi^-\frac{df^-}{f^-}\right)\right]$$

φ^-, f^- on B_j'



for B_j : $\int_{B_j} e_j \frac{df^-}{f^-}$

on B_j' f^- goes from $\theta(b+s)$ to $\theta(b+s+\tau e_j)$

$$\theta(b+s+\tau e_j) = \exp(-\pi i\, e_j^* \tau e_j - 2\pi i\, e_j^*(b+s))\,\theta(b+s)$$

$$\int_{B_j}\frac{df^-}{f^-} = -\pi i\, e_j^*\tau e_j - 2\pi i\, e_j^* b - 2\pi i\, e_j^* s + \mathbb{Z}$$

constant : independent of p & b.

j-component of b

Similarly for A_j-part

$$\int_{A_j}(\cdots\cdots) = \text{constant} + \mathbb{Z}\,\tau e_j$$

Hence $\frac{1}{2\pi i}\int_{\partial(\text{polygon})}\varphi\frac{df}{f} = -b + \text{constant} + \mathbb{Z}^g + \tau\mathbb{Z}^g \equiv -b + \Delta$ (in $J(M)$)

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Another part : $\exists\ \kappa$ (another Riemann vector)

$\Rightarrow$ zeros of $\theta(z - b + \kappa)$ is exactly $W_{g-1}+b$

By combining with the above theorem $\Rightarrow$ analyze $W_1 \cap (W_{g-1}+b)$

This is the theta function part needed in Torelli