More on the Jacobian variety and Torelli theorem

ref: FK. §III.11, III.12

complex torus

$\Pi^{(1)}, \cdots, \Pi^{(2n)}$: $2n$ $\mathbb{R}$-linearly independent vectors in $\mathbb{C}^n$) data

$\Gamma \cong \mathbb{Z}^{2n} \curvearrowright \mathbb{C}^n$ by translation

$\{n_k\}_{k=1}^{2n}$ $\quad z \longmapsto z + \sum_{k=1}^{2n} n_k \Pi^{(k)}$

<u>defn</u> $T = \mathbb{C}^n/\Gamma$ is called a complex torus

Denote the quotient map by ρ

$\rho : \mathbb{C}^n \to T$

1° $\Pi^{(1)}, \cdots, \Pi^{(2n)}$ are $\mathbb{R}$-linearly independent

$\Leftrightarrow$ $\sum_{k=1}^{2n} a_k \Pi^{(k)} = 0$ for $\{a_k\}_{k=1}^{2n} \in \mathbb{R}$ if and only if $a_k = 0$ $\forall k$

e.g. $n=1$ $\quad \Pi^{(1)} = 1$, $\Pi^{(2)} = i$ $\quad \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$

$\Leftrightarrow$ $\begin{bmatrix} \Pi \\ \overline{\Pi} \end{bmatrix}$: $2n \times 2n$ $\mathbb{C}$-matrix is nonsingular

pf: $\{a_k + i b_k\}_{k=1}^{2n} \in$ Kernel : $\sum_{k=1}^{2n} (a_k + i b_k) \Pi^{(k)} = 0 = \sum^{2n} (a_k + i b_k) \overline{\Pi}^{(k)}$

$\Leftrightarrow$ $\sum a_k \Pi^{(k)} = 0 = \sum b_k \Pi^{(k)}$ $\Rightarrow$ $a_k = 0 = b_k$ #

2° Let $u_1, \cdots, u_n$ be the (complex) coordinate for $\mathbb{C}^n$

Then, $du_1, \cdots, du_n$ are holomorphic differential on $\mathbb{C}^n$

and descends as the basis for the space of all holomorphic differentials on T

pf: topology $H_1(T;\mathbb{Z}) \cong \mathbb{Z}^{2n}$. generated by a_k

ρ(path from 0 to $\Pi^{(k)}$)

$\int_{a_k} du_j = \Pi_{jk}$

$= j$-th component of $\Pi^{(k)}$

By 1°, if $\sum_{j=1}^{n} c_j du_j = 0 \Rightarrow c_j = 0$

$\left(\sum_{j=1}^{n} c_j \Pi_{jk} = 0 \Rightarrow \sum_{j=1}^{n} \bar{c}_j \overline{\Pi}_{jk} \Rightarrow [c_1 \cdots c_n \ \bar{c}_1 \cdots \bar{c}_n] \begin{bmatrix} \Pi \\ \overline{\Pi} \end{bmatrix} = 0 \right)$

On the other hand, if δ is a holomorphic differential

$\exists \{c_j\} \in \mathbb{C}^n$ such that $\mathrm{Re} \int_{a_k} \sum_{j=1}^{n} c_j du_j = \mathrm{Re} \int_{a_k} \delta$

$\left([c_1 \cdots c_n \ \bar{c}_1 \cdots \bar{c}_n] \begin{bmatrix} \Pi \\ \overline{\Pi} \end{bmatrix} = \left[\int_{a_1} \delta + \bar{\delta} \ \cdots \ \int_{a_{2n}} \delta + \bar{\delta} \right] \Rightarrow \mathrm{Re} \int_{a_k} \sum_{j=1}^{n} c_j du_j = \mathrm{Re} \int_{a_k} \delta \right)$

$\Rightarrow \exists f : T \to \mathbb{R}$ such that $df = \mathrm{Re}(\delta - \sum_{j=1}^{n} c_j du_j)$

f : harmonic (on n-dim'l torus) $\Rightarrow f =$ constant

$\mathrm{Re}(\delta - \sum_{j=1}^{n} c_j du_j) = 0$. By Cauchy-Riemann, $\mathrm{Im}(\delta - \sum_{j=1}^{n} c_j du_j) = 0$ ※

3° V : (compact, connected) n-dim'l complex manifold

(i.e. local model is open subset of $\mathbb{C}^n$. transition is holomorphic

namely, each component satisfies the Cauchy-Riemann equation)

$\Phi : V \to T = \mathbb{C}^n/\Gamma$ holomorphic

locally

$$\begin{tikzcd} & \mathbb{C}^n \arrow[d, "\rho"] \\ V \arrow[ur, "\tilde{\Phi}"] \arrow[r, "\Phi"'] & T \end{tikzcd}$$

$P_0 \in V$

$$\tilde{\Phi}(P) - \tilde{\Phi}(P_0) = \left(\int_{\tilde{\Phi}(P_0)}^{\tilde{\Phi}(P)} du_1, \cdots, \int_{\tilde{\Phi}(P_0)}^{\tilde{\Phi}(P)} du_n\right)$$

$$= \left(\int_{P_0}^{P} \underline{\tilde{\Phi}^* du_1}, \cdots \int_{P_0}^{P} \underline{\tilde{\Phi}^* du_n}\right)$$

pull-back by $\tilde{\Phi}$

pass to the quotient

$$\Rightarrow \quad \Phi(P) = \Phi(P_0) + \rho\left(\int_{P_0}^{P} \Phi^* du\right)$$

$(\Phi^* du_1, \cdots, \Phi^* du_n)$

4° Consider $T_1 = \mathbb{C}^m/\Gamma_1 \qquad T_2 = \mathbb{C}^n/\Gamma_2$

If $\Phi : T_1 \to T_2$ holomorphic

Φ is determined by $\Phi(P_0)$ & $(\Phi^* du_1, \cdots, \Phi^* du_n)$ $\qquad [A_{j\ell}] : n \times m$

But $\Phi^* du_j$ is a holomorphic differential, $\Phi^* du_j = \sum_{\ell=1}^{m} A_{j\ell}\, dv_\ell$

Hence, up to a translation, Φ is governed by

$$\begin{tikzcd} \mathbb{C}^m \arrow[r, "A"] \arrow[d, "\rho_1"'] & \mathbb{C}^n \arrow[d, "\rho_2"] \\ T_1 \arrow[r, "\Phi(P)-\Phi(P_0)"] & T_2 \end{tikzcd}$$

Prop the only holomorphic map between complex tori are group homomorphisms composed with translations

5° equivalence? $\quad m = n$

$$\mathbb{C}^n/\Gamma \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n}$$

$$\mathbb{C}^n \ni \Pi x \longleftrightarrow x \in \mathbb{R}^{2n}$$

Π: $n \times 2n$

$$\Pi x + \sum_{k=1}^{2n} n_k \Pi^{(k)} \longleftrightarrow x + \sum_{k=1}^{2n} n_k \vec{e}_k$$

Suppose the equivalence is induced by

$A : \mathbb{C}^n \to \mathbb{C}^n$

$$\Rightarrow \quad \begin{tikzcd} \mathbb{R}^{2n} \arrow[r, "M"] \arrow[d, "\Pi_1"'] & \mathbb{R}^{2n} \arrow[d, "\Pi_2"] \\ \mathbb{C}^n \arrow[r, "A"] & \mathbb{C}^n \end{tikzcd} \qquad \exists M \in GL(2n; \mathbb{Z})$$

preserves lattices

$$A\, \Pi_1 = \Pi_2 M$$

A: $GL(n;\mathbb{C})$; Π_1, Π_2: $n \times 2n$

Jacobian variety

0° M: compact Riemann surface, with genus $g > 1$

Fix $2g$ loops-cut $\{A_1, \cdots, A_g, B_1, \cdots, B_g\}$

$\rightsquigarrow \{\zeta_1, \cdots, \zeta_g\}$ basis for holomorphic differential

$$\left[\int_{A_k} \zeta_j \quad \int_{B_k} \zeta_j\right] = [I, \Pi]$$

Π: symmetric with positive definite imaginary part

$$\rightsquigarrow \varphi : M \to J(M) = \mathbb{C}^g / L(M)$$

$P_0 \in M$

1° (universal property)

If $\Phi : M \to T = \mathbb{C}^n/\Gamma$: a holomorphic map

Φ is determined by $(\Phi^* du_1, \cdots, \Phi^* du_n) = \left(\sum_{k=1}^{g} a_{1k}\zeta_k, \cdots, \sum_{k=1}^{g} a_{nk}\zeta_k\right)$

$[a_{jk}]$: $n \times g$ matrix

Consider

$$\begin{tikzcd} \mathbb{C}^g \arrow[r] & \mathbb{C}^n \\ z=(z_1,\dots,z_g) \arrow[r, mapsto] \arrow[d] & az = \left(\sum_{k=1}^g a_{1k}z_k, \dots, \sum_{k=1}^g a_{nk}z_k\right) \arrow[d] \\ \mathbb{C}^g/J(M) \arrow[r, "\psi"] & T = \mathbb{C}^n/\Gamma \end{tikzcd} \Rightarrow \rho(az) + \Phi(P_0)$$

$$\left(\begin{aligned} \Phi^* du_1 &= \sum_{k=1}^g a_{1k}\zeta_k \\ \int_{A_\ell} \Phi^* du_1 &= a_{1\ell} \\ \int_{B_\ell} \Phi^* du_1 &= \sum_{k=1}^g a_{1k}\pi_{\ell k} \end{aligned} \qquad \begin{aligned} &\mathbb{C}^g \longrightarrow \mathbb{C}^n \\ &z+e_{(k)} \longmapsto az + \left(\int_{A_\ell}\Phi^* du = \int_{\Phi(A_\ell)} du\right) \\ &z+\pi^{(\ell)} \longmapsto az + \left(\int_{B_\ell}\Phi^* du = \int_{\Phi(B_\ell)} du\right) \end{aligned}\right)$$

$$\Rightarrow \exists\, \psi : J(M) \to T \quad \Rightarrow \quad \begin{tikzcd} J(M) \arrow[dr, "\psi"] & \\ M \arrow[u, "\varphi"] \arrow[r, "\Phi"] & T \end{tikzcd} \qquad \Phi = \psi \circ \varphi$$

Symmetric product / integral divisors

1° $M_n = \{ D \in \mathrm{Div}(M) \mid \deg D = n,\ (D) \geq 1\}$

$M_n \xrightarrow{\varphi} W_n \subset J(M)$ Jacobi inversion: $W_g = J(M)$

Set $W_0 = \{0\}$

Since $\varphi(DP_0) = \varphi(D)$, $W_n \subset W_{n+1}$

goal study the structure of W_n

$D \in M_n$ $\quad r(D^{-1}) = i(D) + n + 1 - g \qquad$ defn $W_n^r = \varphi(\{D \mid r(D^{-1}) \geq r+1\})$

e.g. Canonical (& integral) divisor $\quad \deg D = 2g-2$

$D = (\omega) \iff i(D) \geq 1$ (in fact, "=" since $\deg D = 2g-2$)

$r(D^{-1}) \geq 1 + 2g-2+1-g$
$= g = (g-1)+1$

$\Rightarrow W_{2g-2}^{g-1} = \{K\}$

$\leftarrow$ any two canonical divisors differ by a principal divisor $\Rightarrow$ has the same image under φ

2° structure of M_n

$M_n = M \times M \times \cdots \times M \,/\, S_n \to$ symmetric group of n-elements

(set theoretically)

$$\begin{tikzcd} M^n \arrow[d, "p"] & \\ M_n \arrow[r, "f"] & \mathbb{C} \end{tikzcd}$$

quotient map $\to p$

defn f is said to be continuous (holomorphic) if $f \circ p$ is continuous (holomorphic) on M^n

Given $(P_1, \dots, P_n) \in M^n \longleftrightarrow D = P_1 \cdots P_n \in M_n$

simplest case $\quad P_i \neq P_j \ \forall i \neq j \quad$ choose $P_i \in (\mathcal{U}_i, z_i)$, $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$

$f \circ p \longleftrightarrow$ function on $\mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n$

use $S_n \curvearrowright$ on $\mathcal{U}_{\sigma(1)} \times \mathcal{U}_{\sigma(2)} \times \cdots \times \mathcal{U}_{\sigma(n)}$

Since σ acts biholomorphically.

it suffices to check $f \circ p$ is holomorphic on $\mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n$

<u>in general</u> $D = Q_1^{m_1} \cdots Q_s^{m_s}$ choose $Q_i \in (\mathcal{U}_i, z_i)$ $\left(\sum_{i=1}^{n} m_i = n\right)$

$f \circ p \longleftrightarrow$ function on $\underbrace{\mathcal{U}_1 \times \cdots \times \mathcal{U}_1}_{m_1} \times \underbrace{\mathcal{U}_2 \times \cdots \times \mathcal{U}_2}_{m_2} \times \cdots \times \underbrace{\mathcal{U}_s \times \cdots \times \mathcal{U}_s}_{m_s}$

which is invariant under $S_{m_1} \times \cdots \times S_{m_s}$

focus on $\underset{(z_1 \cdots\cdots z_m)}{\mathcal{U} \times \cdots \times \mathcal{U}} \circlearrowleft S_m$ where $\{0\} \in \mathcal{U} \subset \mathbb{C}$

f: invariant under $S_m \Rightarrow$ the power series of $f(z_1, \cdots, z_m)$ must be power series in $\zeta_1 = (-1)\sum_{j=1}^{m} z_j$, $\zeta_2 = (-1)^2 \sum_{j<k} z_j z_k$, $\cdots$, $\zeta_m = (-1)^m z_1 \cdots z_m$

or equivalently, $t_k = \sum_{j=1}^{m} z_j^k$

Also, note that solutions of $z^m + \zeta_1 z^{m-1} + \cdots + \zeta_m = 0$ are exactly the (un-ordered) set $\{z_1, \cdots, z_m\}$

$\Rightarrow$ $(\zeta_1, \cdots, \zeta_m)$ (also $(t_1, \cdots, t_m)$) is the holomorphic coordinate on M_n

e.g. $M_3 = M \times M \times M / S_3$ $\quad f: M \to \mathbb{C} \rightsquigarrow$

$$\begin{tikzcd} M \times M \times M \arrow[d, "p"'] \arrow[dr, "\widetilde{F} = f+f+f"] & \\ M_3 \arrow[r, "F"'] & \mathbb{C} \end{tikzcd}$$

i) P^3 $\quad P \in (\mathcal{U}, z)$

$\mathcal{U} \times \mathcal{U} \times \mathcal{U}$ is a nbd of $\underset{z_1, z_2, z_3}{(P, P, P)} \in M \times M \times M$, $\circlearrowleft S_3$

$$\widetilde{F}(z_1, z_2, z_3) = f_u(z_1) + f_u(z_2) + f_u(z_3)$$

f_u: local expression of $f|_u = \sum_{n=0}^{\infty} a_n z^n$

$$\Rightarrow \widetilde{F}(z_1, z_2, z_3) = 3a_0 + a_j \sum_{j=1}^{\infty} (z_1^j + z_2^j + z_3^j)$$

$$= 3a_0 + a_1 t_1 + a_2 t_2 + a_3 t_3 + (\text{higher order})$$

$\Rightarrow F$ vanishes at P^3 if $(f|_u) \geq P^3$

ii) $P^2 Q$ $\quad P \in (\mathcal{U}, z)$ $\quad Q \in (\mathcal{V}, w)$ $\quad P \neq Q \Rightarrow$ may assume $\mathcal{U} \cap \mathcal{V} = \phi$

$\mathcal{U} \times \mathcal{U} \times \mathcal{V} \in M \times M \times M$, $\circlearrowleft S_2$ $\quad$ disjoint from $\mathcal{U} \times \mathcal{V} \times \mathcal{U}$, $\mathcal{V} \times \mathcal{U} \times \mathcal{U}$

$$\widetilde{F}(z_1, z_2, w) = f_u(z_1) + f_u(z_2) + f_v(w)$$

Similarly, F vanishes at $P^2 Q$ if $(f|_u) \geq P^2$, $(f|_v) \geq Q$

3° $\varphi: M \to J(M)$ $\quad$ by construction, $\varphi^* dv_j = \zeta_j$

$\rightsquigarrow \varphi_n: M_n \to J(M)$ induced by

$$\begin{tikzcd} M \times \cdots \times M \arrow[d, "p"'] \arrow[dr, "\widetilde{\varphi}_n = \varphi + \cdots + \varphi"] & \\ M_n \arrow[r, "\varphi_n"'] & J(M) \end{tikzcd}$$

usually, still denote it by φ

<u>lemma</u> δ: holomorphic differential on $J(M)$

$\varphi_n^* \delta$ vanishes at $D \in M_n \iff (\varphi^* \delta) \geq D$

pf: by the same method as above. #

structure of $\varphi: M_n \to W_n \subset J(M)$

$$M_n^r = \{ D \in M_n \mid r(D^{-1}) \geq r+1 \}$$

$W_n^r = \varphi(M_n^r)$ $\quad$ By Abel's theorem, $\varphi^{-1}(W_n^r) = M_n^r$

$\hookrightarrow$ same $\varphi \Rightarrow$ differ by principal divisor

prop a) $D \in M_n$. The Jacobian of $\varphi: M_n \to W_n \subset J(M)$ at D has rank $n+1-r(D^{-1}) = g - i(D)$

($r(D^{-1}) = 1 \iff$ full rank)

b) Let $\nu(D) = r(D^{-1}) - 1$
$\exists$ injective holomorphic map from $\mathbb{P}^{\nu}$ to M_n, whose image is exactly $\varphi^{-1}(\varphi(D))$

c) $\varphi: M_n \setminus M_n' \longrightarrow W_n \setminus W_n'$ is a biholomorphism

rmk $\hat{\mathbb{C}} \to \mathbb{CP}^2$ injective, holomorphic
$z \longmapsto [1; z^2; z^3]$ differential vanishes at $z=0$
$[Z_0; Z_1] \longmapsto [Z_0^3; Z_0 Z_1^2; Z_1^3]$

pf: Examine $\varphi: M_n \to J(M)$ at $D = Q_1^{m_1} \cdots Q_s^{m_s}$ $\begin{cases} m_j > 0 & \sum_{j=1}^{s} m_j = n \\ Q_i \neq Q_j & \forall i \neq j \end{cases}$

$d\varphi_D$: the Jacobian of φ at D can be represented by a $g \times n$ $\mathbb{C}$-matrix

(recall for $n=1$, by construction. $d\varphi = \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_g \end{bmatrix}$)

$Q_i \in (\mathcal{U}_i, z_i)$ $z_i(Q_i) = 0$, $\mathcal{U}_i \cap \mathcal{U}_j = \phi$ $\forall i \neq j$

$$\overbrace{\mathcal{U}_1 \times \cdots \times \mathcal{U}_1}^{m_1} \times \cdots \times \overbrace{\mathcal{U}_s \times \cdots \times \mathcal{U}_s}^{m_s} \subset M^n \xrightarrow[S_{m_1} \times \cdots \times S_{m_s}]{\text{quotient by}} \text{nbd of } D \in M$$

$\circlearrowleft S_{m_1}$ $\circlearrowleft S_{m_s}$

$((z_{11}, \ldots, z_{1m_1}), \ldots, (z_{s1}, \ldots, z_{sm_s}))$

$1 \le \mu \le g$ $\zeta_\mu \big|_{\mathcal{U}_j} = \left(\sum_{\ell=0}^{\infty} a_\ell^{\mu j} (z_j^\ell) \right) dz_j = d\left(\sum_{\ell=0}^{\infty} \frac{1}{\ell+1} a_\ell^{\mu j} (z_j)^{\ell+1} \right)$

$\rightsquigarrow \sum_{\ell=0}^{\infty} \sum_{i=1}^{m_j} \frac{1}{\ell+1} a_\ell^{\mu j} (z_{ji})^{\ell+1} = \sum_{\ell=1}^{m_j} \frac{1}{\ell} a_{\ell-1}^{\mu j} t_{j\ell} + \left(\text{higher order terms}\right)$

$t_{j\ell} = \sum_{i=1}^{m_j} (z_{ji})^\ell$

$((t_{11}, \cdots t_{1m_1}), \cdots (t_{s1}, \cdots, t_{sm_s}))$ is the coordinate on M_n

$$\Rightarrow d\varphi\big|_D = \begin{bmatrix} a_0^{11} & \frac{1}{2} a_1^{11} & \cdots & \frac{1}{m_1} a_{m_1-1}^{11} & \cdots & a_0^{1s} & \cdots & \frac{1}{m_s} a_{m_s-1}^{1s} \\ a_0^{21} & \frac{1}{2} a_1^{21} & & \frac{1}{m_1} a_{m_1-1}^{21} & \cdots & & \ddots & \\ & & \ddots & \vdots & & & & \\ a_0^{g1} & \frac{1}{2} a_1^{g1} & \cdots & \frac{1}{m_1} a_{m_1-1}^{g1} & & a_0^{gs} & \cdots & \frac{1}{m_s} a_{m_s-1}^{gs} \end{bmatrix}$$

$$= \begin{bmatrix} \zeta_1(Q_1) & \cdots & \frac{1}{m_1!} \zeta_1^{(m_1-1)}(Q_1) & \zeta_1(Q_s) & \cdots & \frac{1}{m_s!} \zeta_1^{(m_s-1)}(Q_s) \\ & \ddots & & \cdots & \ddots & \vdots \\ \zeta_g(Q_1) & \cdots & \frac{1}{m_1!} \zeta_1^{(m_1-1)}(Q_1) & \zeta_g(Q_s) & \cdots & \frac{1}{m_s!} \zeta_g^{(m_s-1)}(Q_s) \end{bmatrix}$$

rank = rank (transpose) = $g -$ ker (transpose)
$= g - i(D)$

for part (c) : full-rank, also by Abel's theorem.

for part (b) : Abel : $\varphi(D') = \varphi(D) \Rightarrow D' D^{-1} = (f)$

$\Rightarrow D' = (f) D$ $\qquad (D') \geq 1 \Leftrightarrow (f) \geq D^{-1}$

$r(D^{-1}) = \dim L(D^{-1}) = \nu + 1$

choose a basis : $\{f_0, \cdots, f_\nu\}$ for $L(D^{-1})$

Consider $0 \neq (c_0, \cdots, c_\nu) \xrightarrow{\Phi} \underbrace{D(\sum_{j=0}^{\nu} f_j)}_{\text{this divisor}} \in M_n$

Not hard to see that $\Phi(c) = \Phi(\tilde{c}) \Leftrightarrow c = \lambda \tilde{c}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$

holomorphicity of Φ : by argument principle ... ※

rmk $\quad d\varphi|_D : \mathbb{C}^n \to \mathbb{C}^g$

$\rightsquigarrow$ denote the vectors here by $\sum_{\mu=1}^{g} c_\mu \frac{\partial}{\partial u_\mu}$

$\updownarrow$ natural pair with $\sum_{\mu=1}^{g} b_\mu du_\mu$ $\qquad \sum_{\mu=1}^{g} c_\mu b_\mu$

The above matrix representative of $d\varphi|_D$ shows that $\sum_{\mu=1}^{g} c_\mu \frac{\partial}{\partial u_\mu} \in \text{image}(d\varphi|_D)$

$\Leftrightarrow \sum_{\mu=1}^{g} c_\mu b_\mu = 0$ for all $\sum_{\mu=1}^{g} b_\mu \zeta_\mu \in \Omega(D)$

$\updownarrow$ under φ^*

$\sum_{\mu=1}^{g} b_\mu du_\mu$

$\left(c_\mu = \sum_{i=1}^{n} T^i_\mu v_i \qquad \sum_{\mu=1}^{g} b_\mu \zeta_\mu \in \Omega(D) \Leftrightarrow \sum_{\mu=1}^{g} T^i_\mu b_\mu = 0 \cdots \right)$

In other words, image $(d\varphi|_D)$ = Annihilator ($\Omega(D)$ by φ^*)

Cor $\quad W_n \subset J(M)$ is irreducible i.e. if $h : J(M) \to \mathbb{C}$ is meromorphic, and $h|_{W_n}$ vanishes on an open subset $\Rightarrow h \equiv 0$ on W_n

pf: Consider $h \circ \varphi$ on M_n $\leftarrow$ compact, connected complex manifold. ※

not an example $\quad \{u_1 = 0\} \cup \{u_2 = 0\} \subset \mathbb{C}^2$ is not irreducible

More properties on W_n $\quad n \in \{0, 1, \cdots, g-1\}$

def for $S \subset J(M)$, $a \in J(M)$

define $\quad S + a = \{ s + a \mid s \in S \}$ (shift S by a)

$-S = \{ -s \mid s \in S \}$ (inverse elements of a)

for $S, T \subset J(M)$

$S \oplus T = \bigcup \{ S + t \mid t \in T \} = \{ s + t \mid s \in S, t \in T \}$

$S \ominus T = \bigcap \{ S - t \mid t \in T \}$

prop $\forall a \in J(M)$

$-W_{g-1} - a = W_{g-1} - a - K$ → image of a canonical divisor

pf: All are the same as the $a=0$ case.

For $D \in M_{g-1}$ ⇝ $(g-1)$ linear condition on $\mathbb{C}^g$ = holo diff on M

$\Rightarrow \exists (\omega) \geq D$. Hence, $(\omega) = D'D$, $(D') \geq 1$

Since $\deg(\omega) = 2g-2$, $\deg(D) = g-1 \Rightarrow \deg D' = g-1$

$\varphi(D) + \varphi(D') = K \Rightarrow -W_{g-1} = W_{g-1} - K$ ※

prop $0 \leq r \leq t \leq g-1$, $a \in J(M)$, $b \in J(M)$. Then,

$(W_r + a) \subset (W_t + b) \iff a \in (W_{t-r} + b) \iff b \in (-W_{t-r} + a)$

basically by definition.

pf: $\Leftarrow$) direct

$\Rightarrow$) goal find $A \in M_{t-r}$ such that $a = \varphi(A) + b$

For any $D \in M_r$. $\exists D' \in M_t$

$\ni \varphi(D) + a = \varphi(D') + b$

By taking $D = P_0$ $\Rightarrow \exists B \in M_t \ni a = \varphi(B) + b$

Let $(A) \geq 1$ and $a = \varphi(A) + b$ with minimal degree $\overset{\shortparallel}{t}$

• $r(A^{-1}) = 1$: if $r(A^{-1}) > 1$

$\Rightarrow r(A^{-1}P_0) > 0 \Rightarrow \exists f \in L(A^{-1}P_0)$

$\Rightarrow (f) \geq A^{-1}P_0$

$\Rightarrow (f)AP_0^{-1} \geq 0$ $\quad (f)AP_0^{-1} \sim AP_0^{-1}$

$\overset{\varphi}{\leadsto} \varphi(A)$ same φ. lower degree $\rightarrow\leftarrow$

Now, for any $D \in M_r$, $\exists D' \in M_t$

$\varphi(D) + \varphi(A) = \varphi(D')$ $\quad \deg A \leq t$

Recall that $\varphi(M_{r'}) \subset \varphi(M_r)$ if $r \leq r'$

If $\deg A > t-r$, $\quad t+1-\deg A \leq r$

Choose D of degree $t+1-\deg A$ ($\rightarrow$ can apply assumption on D)

such that $r(A^{-1}D^{-1}) = 1$ and $P_0 \notin D$

By R-R & $r(A^{-1}) = 1$, $r(A^{-1}D^{-1}) = 1$

$\iff i(AD) = i(A) - \deg D$

Since $\deg A + \deg D \leq g$. $\exists$ such D with $P_0 \notin D$

$\Rightarrow$ By assumption, $\varphi(D) + \varphi(A) = \varphi(D')$ for some $D' \in M_t$

$\varphi(AD(D')^{-1}) = 0$

By Abel, $AD(D')^{-1}P_0^{-1} \sim$ principal divisor

$AD(D')^{-1}P_0^{-1} = (f) \Rightarrow f^{-1} \geq A^{-1}D^{-1}$ $\quad f^{-1} \in L(A^{-1}D^{-1})$ is $\mathbb{C}$

$\Rightarrow f =$ constant, $AD = D'P_0$ but $P_0 \notin A$, $P_0 \notin D$ $\rightarrow\leftarrow$ ※

rmk if $W_r + a \subset W_r \Rightarrow a \in W_0 + 0 = 0$

$\Rightarrow a = 0$

That is to say. W_r is very far from being translation invariant.

prop i) $(W_r+a)\oplus(W_t+b) = W_{s+t} + a+b$

ii) $0\le r\le t \le g-1$. $a,b\in J(M)$

$(W_t+a)\ominus(W_r+b) = W_{t-r}+a-b$

pf: i) straight forward

ii) $u\in W_{t-r}+a-b \iff b\in W_{t-r}-u+a$

$\iff W_r+b\subset W_t-u+a$

i.e. for any $v\in W_r$. $u\in(W_t+a)-(v+b) \underset{\Leftarrow}{\Rightarrow} u\in(W_t+a)\ominus(W_r+b)$ #

rmk $a=b=0$ $t=g-1$, $r=g-2$ $\Rightarrow$ $W_{g-1}\ominus W_{g-2}=W_1 \overset{\varphi}{\cong} M$

M can be reconstructed from W_{g-1} & W_{g-2}

Focus on W_{g-1}

Cor $0\le r\le g-1$

i) $(W_r+a)\le(W_{g-1}+b) \iff a\in W_{g-1-r}+b$

ii) $W_{g-1-r} = W_{g-1}\ominus W_r$

iii) $-W_{g-1-r}+K = W_{g-1}\ominus(-W_r)$

pf: i) & ii) : special case of above prop.

iii) $W_{g-1}\ominus(-W_r) = \bigcap\{W_{g-1}+u \mid u\in W_r\}$

$= \bigcap\{-W_{g-1}+u+K \mid u\in W_r\}$

$= \bigcap\{-W_{g-1}+u \mid u\in W_r\}+K$

$= -(W_{g-1}\ominus W_r)+K$

$= -W_{g-1-r}+K$ #

Torelli's theorem

Lemma $r\in\{0,1,\cdots,g-2\}$. $a,b\in J(M)$ such that

$b=a+x-y$ for some $x\in W_1$, $y\in W_{g-1-r}$

Then, either $(W_{r+1}+a)\subset(W_{g-1}+b)$

or $(W_{r+1}+a)\cap(W_{g-1}+b) = (W_r+a+x)\cup\big((W_{r+1}+a)\cap(-W_{g-2}-y+a+K)\big)$

pf: $x=\varphi(\overset{\in M}{P})$ $y=\varphi(\overset{\in M_{g-1-r}}{D})$ $\in M_{g-2-r}$

If $P\in D$. $b=a-\varphi(DP^{-1})$

$\Rightarrow a\in W_{g-2-r}+b$

By the prop. $(W_{r+1}+a)\subset(W_{g-1}+b)$

Now, suppose that $P\notin D$.

For $u\in(W_{r+1}+a)\cap(W_{g-1}+b)$. $\exists D'\in M_{r+1}$ and $D''\in M_{g-1}$

such that $u=\varphi(D')+a=\varphi(D'')+b$

$\Rightarrow \varphi(D')=\varphi(D'')-\varphi(D)+\varphi(P) \Rightarrow D'D\sim D''P$

<u>case 1</u> $D'D = D''P$: Since $P \notin D \Rightarrow P \in D'$

$$u = \varphi(D') + a = \varphi(\underset{\in M_r}{D'P^{-1}}) + \varphi(P) + a = \varphi(DP^{-1}) + x + a$$
$$\in W_r + x + a$$

<u>case 2</u> $D'D \neq D''P$: $\exists f$: non-constant meromorphic function such that $(f) = D''P(D'D)^{-1} \geq (D'D)^{-1} \Rightarrow r((D'D)^{-1}) > 1$

Hence, $\forall Q \in M$. $\exists h \in L((\underset{r+1}{D'}\underset{g-1-r}{D})^{-1}) \rightarrow (h) \geq Q$

$\Rightarrow \exists \tilde{D} \in M_{g-1}$ such that $Q\tilde{D} \sim D'D$

$$u = \varphi(D') + a = \varphi(\underset{\in M\text{: any}}{Q}) + \varphi(\underset{\in M_{g-1}\text{: exist}}{\tilde{D}}) - \underset{= y}{\varphi(D)} + a$$

for any Q

$\Rightarrow u \in W_{g-1} - y + a + \varphi(Q) \quad \forall Q \in M$

$\Rightarrow u \in \bigcap \{W_{g-1} - y + a + v \mid v \in W_1\}$

$= -\bigcap \{-W_{g-1} + y - a - v \mid v \in W_1\} = -\bigcap \{W_{g-1} - K + y - a - v\}$

$= -(W_{g-1} - K + y - a) \ominus W_1 = -W_{g-2} + K - y + a$

On the other hand.

It is clear that $W_r + a + \overset{\varphi(P)}{x} \subset W_{r+1} + a$

$= W_r + b + \underset{\in W_{g-1-r}}{y} \subset W_{g-1} + b$

Also, $\underset{\varphi(\check{D})}{-(W_{g-2} + y - a) + K} \subset \underset{\varphi(\check{D}P)}{-(W_{g-1} + y - a - x) + K} = W_{g-1} + b$ ※

<u>observation</u> $\{A_1, \cdots, A_g, B_1, \cdots, B_g\}$ 2g-loops-cut as canonical basis for homology. if we switch it to $\{-A_1, \cdots, -A_g, -B_1, \cdots, -B_g\}$

- intersection relation is the same
- $\{\zeta_1, \cdots, \zeta_g\} \longrightarrow \{-\zeta_1, \cdots, -\zeta_g\} \Rightarrow$ same period matrix

Namely, $L(M)$ is exactly the same, but $\varphi \rightarrow -\varphi$ and hence $W_r \rightarrow -W_r$.

<u>fact</u> $\exists$ c (a Riemann constant vector) $\in J(M)$ such that for any $b \in J(M)$, if $W_1 \not\subset (W_{g-1} + b)$ then $W_1 \cap (W_{g-1} + b) = \{\varphi(P_1), \cdots, \varphi(P_g)\}$ (repeated points $\Leftrightarrow$ multiplicity) and $\varphi(P_1 \cdots P_g) - b = c$

<u>thm</u> (Torelli) $g \geq 2$, M is determined by $W_{g-1} \subset J(M)$ up to translation and reflection.

called the canonical polarization

pf: $\varphi : M \longrightarrow J(M)$ assume $\varphi(P_0) = 0 = \psi(Q_0)$

$\qquad\qquad\qquad\quad\ \ \cong$

$\psi : N \longrightarrow J(N)$ Denote by $\psi(N_r)$ by V_r

The theorem says that if $V_{g-1} = W_{g-1} + a \Rightarrow V_1$ is a translate of W_1 or $-W_1$.

Since $V_{g-1} = W_{g-1} + a \Rightarrow V_1 \subset W_{g-1} + a$

claim : $\exists\ s \in J(M)$, $Q \neq Q' \in N$, $x \neq x' \in W_1 = J(M)$

such that $\psi(Q) = x + s$ i.e. $V_1 \cap (W_1 + s)$

$\psi(Q') = x' + s$ has two distinct points.

Since $x \neq x'$, $W_{g-1} - x \not\subset W_{g-1} - x'$ ($\Leftrightarrow$ $-x \in W_0 - x'$: NOT True)

by the lemma, $(W_{g-1} - x) \cap (W_{g-1} - x') = W_{g-2} \cup \left(\begin{array}{l} (W_{g-1} - x) \\ \cap(-W_{g-2} - x - x' + K) \end{array} \right)$

$(a = -x,\ b = -x',\ y = a + x - b = x')$

Also, $W_{g-2} + x' \subset W_{g-1}$

$\Leftrightarrow -W_{g-2} - x - x' + K \subset -W_{g-1} + x + K = W_{g-1} - x$

Hence, $(W_{g-1} - x) \cap (W_{g-1} - x') = W_{g-2} \cup (-W_{g-2} - x - x' + K)$

$= (V_{g-1} - (a+s) - \psi(Q)) \cap (V_{g-1} - (a+s) - \psi(Q'))$

by lemma $= (V_{g-2} + *) \cup (\cdots)$

!

$\Rightarrow V_{g-2} + * \subset W_{g-2}$ or $(-W_{g-2} - x - x' + K)$

Since $W_1 = W_{g-1} \ominus W_{g-2}$, $-W_1 + K = W_{g-1} \ominus (-W_{g-2})$

$= V_{g-1} \ominus W_{g-2} + a$ similarly, $V_1 \subset -W_1 + (\text{translate})$

$\supset V_{g-1} \ominus V_{g-2} + a - *$

$\supset V_1 + a - *$ ※

For the claim : need to use the lemma & the fact