uniformization theorem

$M \overset{\text{biholomorphic}}{\cong} \begin{cases} \hat{\mathbb{C}} & g=0 \quad (\text{Riemann-Roch}) \\ \mathbb{C}/\text{lattice} & g=1 \quad (\text{Abel-Jacobi}) \\ ? & g>1 \end{cases}$

<u>recall</u> Riemann mapping $\Omega \subset \mathbb{C}$ open & simply-connected, $\Omega \neq \mathbb{C} \Rightarrow \Omega \cong D \cong \mathbb{H}$

<u>plan</u> given M. $\exists \tilde{M}$. quotient by discrete group action.

$\tilde{M} \downarrow M$ $\quad \tilde{M}$: simply-connected $\Rightarrow \tilde{M} \cong D$ if $g>1$

Focus on simply-connected Riemann surface. (may be non-compact)

digression on Aut(D) & Aut($\hat{\mathbb{C}}$)

$f: D \to D$, $\quad z \mapsto \dfrac{z-a}{1-\bar{a}z}$ $\quad |a|<1$

$f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ $\quad (\mathbb{C} \subset \hat{\mathbb{C}})$, $\quad z \mapsto \dfrac{az+b}{cz+d}$ $\quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$

holomorphic function $\rightsquigarrow$ only need real part $\rightsquigarrow$ solve imaginary part by Cauchy-Riemann

$f \rightsquigarrow \log f$: easier to describe the boundary value for its real part (at least, for Aut(D), $\mathrm{Re}(\log f)|_{\partial D} = 0$)

Re log

$\log|z-a|$ + harmonic $\qquad \log|z+\frac{b}{a}| - \log|z+\frac{d}{c}|$ + harmonic

$\rightsquigarrow$ study harmonic function with log singularity

harmonic & subharmonic function [Ahlfors, ch.6.3] [Gamelin, ch. XV]

$\Omega \subset M$: open set on a Riemann surface

<u>harmonic</u> $u \in \mathcal{C}(\Omega; \mathbb{R})$, $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+\rho e^{i\theta})\, d\theta$

Poisson : $u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z|^2}{|\rho e^{i\theta} - z|^2} \underline{u(\rho e^{i\theta})}\, d\theta$ $\to$ only on the boundary

- strong maximum principle. if u attains maximum in the interior $\Rightarrow u$ must be a constant
- Harnack inequality K: compact set $\exists\, C>0$ such that $\frac{1}{C} \le \frac{u(p)}{u(q)} \le C$ $\forall p, q \in K$, u: positive harmonic function
- Harnack principle, $\{u_n\}$: harmonic, non-decreasing $\Rightarrow u_n \to u$: either harmonic or ∞ and uniformly on compact sets

<u>subharmonic</u> $v \in \mathcal{C}(\Omega; \mathbb{R})$, $v(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} v(z_0+\rho e^{i\theta})\, d\theta$

$\Leftrightarrow \forall\, \Omega' \subset \Omega$ $\; u$: harmonic on Ω', $v-u$ satisfies the strong maximum principle

- if v: $\mathcal{C}^\infty$ and $\Delta v \ge 0 \Rightarrow v$ is subharmonic

prop if v_1 & v_2 : subharmonic, then $\max(v_1, v_2)$ is subharmonic

pf: $v = \max(v_1, v_2)$

Given u: harmonic on Ω'

if $\max(v-u)$ happens at z_0, and $v(z_0) = v_1(z_0)$

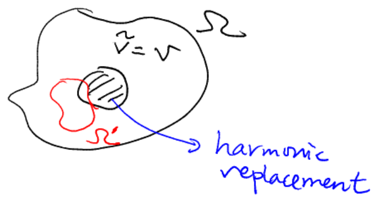
$\Rightarrow$ $(v_1-u)(z) \le (v-u)(z) \le (v-z)(z_0) = (v_1-u)(z_0)$

$\Rightarrow$ $v_1 - u \equiv$ constant on Ω' & all the equality holds

$\Rightarrow$ $v-u$: strong max principle. ※

prop v: subharmonic. given $\overline{B(z;\rho)} \subset \Omega$

define $\tilde{v}$ by $\begin{cases} v & z \in \Omega \setminus B(z;\rho) \\ \text{Poisson} & z \in B(z;\rho) \end{cases}$



Then, $\tilde{v}$ is also subharmonic (note that $\tilde{v} \ge v$)

pf: Given u: harmonic on Ω', if $\max(\tilde{v}-u)$ happens at z_0

$(\tilde{v}-u)(z) \le (\tilde{v}-u)(z_0)$

1° $z_0 \notin \overline{B(z;\rho)}$

$(v-u)(z) \le (\tilde{v}-u)(z) \le (\tilde{v}-u)(z_0) = (v-u)(z_0)$ ......

2° $z_0 \in B(z;\rho)$, harmonic $\Rightarrow$ $(\tilde{v}-u)(z) \equiv (\tilde{v}-u)(z_0)$ $\forall z \in \Omega' \cap B(z;\rho)$

$\Rightarrow$ max also happens at $\Omega' \cap \partial B(z;\rho)$ $\to$ back to 1° ※

defn $\mathfrak{F}$: a family of subharmonic functions is called <u>Perron's family</u>

if $\begin{cases} v_1, v_2 \in \mathfrak{F} \Rightarrow \max(v_1, v_2) \in \mathfrak{F} \\ v \in \mathfrak{F}, \overline{B(z;\rho)} \subset \Omega \Rightarrow \tilde{v} \in \mathfrak{F} \end{cases}$ (harmonic replacement on $B(z;\rho)$)

thm $\mathfrak{F}$: Perron's family, let $u(z) = \sup\{v(z) \mid v \in \mathfrak{F}\}$

Then, either u is harmonic, or $u \equiv \infty$

sketch of the proof: 1° Fix $z_0 \in \Omega$, choose $\overline{B(z_0;\rho)} \subset \Omega$

$\exists \{v_n\} \in \mathfrak{F}$ $\to$ $\lim_{n\to\infty} v_n(z_0) = u(z_0)$

$V_n = \max\{v_1, \dots, v_n\} \in \mathfrak{F}$ V_n : non-decreasing and $V_n(z_0) \to u(z_0)$

$\rightsquigarrow$ $\tilde{V}_n \in \mathfrak{F}$, still non-decreasing & $\tilde{V}_n(z_0) \to u(z_0)$

$\boxed{\text{check}}$

By Harnack principle. $\tilde{V}_n \to U$: harmonic on $B(z_0;\rho)$ and the convergence is uniform on compact subsets.

2° $U \equiv u$ on $B(z_0;\rho)$

Given $z_1 \in B(z_0;\rho) \setminus \{z_0\}$. $\exists\, w_n \in \mathfrak{F}$, $w_n(z_1) \to u(z_1)$

Consider $\bar{w}_n = \max(w_n, V_n)$ subharmonic & has the desired limit at z_0 & z_1

$\rightsquigarrow$ Similarly, consider $W_n = \max\{w_1, \dots, w_n\}$, then $\tilde{W}_n$

By Harnack $\tilde{W}_n \to U_1$ = harmonic on $B(z_0;\rho)$ and $\begin{cases} U_1(z_1) = u(z_1) \\ U_1(z_0) = u(z_0) \end{cases}$

Since $\tilde{V}_n \le \tilde{W}_n$, $U \le U_1 \le u$ on $B(z_0;\rho)$

$U(z_0) = U_1(z_0) \Rightarrow U \equiv U_1$ on $B(z_0;\rho)$

$\Rightarrow U(z_1) = U_1(z_1) = u(z_1)$ $\forall z_1$ ※

$\bar{w}_n \ge V_n$

$\Rightarrow W_n \ge V_n$

$\Rightarrow \tilde{W}_n \ge \tilde{V}_n$

$\Rightarrow U_1 \ge U$

Green's function

Fix $q \in M$. choose a local coordinate on a neighborhood of q $(\mathcal{U}, z)$ with $z(q) = 0$

goal looking for harmonic function with singularity $\log|z|$

Perron family $\leftarrow$ $\mathcal{F}_q = \{ v$ = subharmonic on $M \setminus \{q\}$, $v \equiv 0$ on $M \setminus \{\text{compact set}\}$, $v + \log|z|$ is subharmonic on $\mathcal{U}$. $\}$ $\to$ non-empty, will see it momentarily

defn If the upper envelope (sup) is finite, we say the Green's function for M with pole at q exists, and denote it by

$$g(p,q) = \sup\{ v(p) \mid v \in \mathcal{F}_q \}$$

if the sup is ∞, we say the Green's function does NOT exist

thm If $g(p,q)$ exists, it is harmonic & positive on $M \setminus \{q\}$

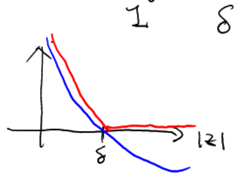
$g(p,q) + \log|z|$ is harmonic on $\mathcal{U}$

Moreover, for any positive harmonic function $h(p)$ with $h(p) + \log|z|$ being harmonic on $\mathcal{U}$, $h(p) \geq g(p,q)$ $\forall p \in M \setminus \{q\}$

pf: 0° By construction, $g(p,q)$: harmonic on $M \setminus \{q\}$ and $g(p,q) \geq 0$

1° $\delta < \rho$. $-\log|z| + \log\delta \geq 0$, harmonic $\Rightarrow$ subharmonic $\in \mathcal{F}_q$

$\hookrightarrow$ extend it by 0 on $M \setminus \{|z(p)| < \delta\}$



$\Rightarrow g(p,q) \geq -\log|z(p)| + \log\delta$ when $|z(p)| < \delta$

$\Rightarrow g(p,q) > 0$ when $|z(p)| < \delta$, $p \neq q$

By the strong min principle $g(p,q) > 0$ on $M \setminus \{q\}$

2° ($g(p,q) + \log|z(p)|$: harmonic on $\mathcal{U}$) Prove boundedness

let $M = \max_{|z(p)|=\delta} g(p,q)$ $\Rightarrow |v(p)| \leq M$ $\forall v \in \mathcal{F}_q$, $|z(p)| = \delta$

Since $v(p) + \log|z(p)|$ is harmonic $\Rightarrow v(p) + \log|z(p)| \leq M + \log\delta$

$\Rightarrow g(p,q) + \log|z(p)| \leq M + \log\delta$ $\quad \forall |z(p)| < \delta$, $p \neq q$

$\Rightarrow g(p,q) + \log|z(p)|$ is a bounded harmonic function with singularity @ $p = q$ $\Rightarrow$ removable

3° $h(p) > 0$ & harmonic on $M \setminus \{q\}$, $h(p) + \log|z(p)|$: harmonic at q

$\forall v \in \mathcal{F}_q$. $v - h$ subharmonic on M

$v - h < 0$ outside a compact set K

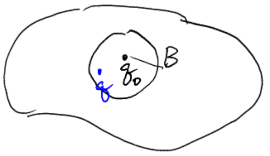
By the strong max principle, $\max_K (v-h)$ cannot ≥ 0

$\Rightarrow v - h < 0 \Rightarrow g(p,q) \leq h(p)$ ※

Cor $\inf\{ g(p,q) \mid p \in M \setminus \{q\} \} = 0$

pf: if $\inf = a > 0$ $\Rightarrow$ Consider $h(p) = g(p,q) - a \geq g(p,q)$ $\to\leftarrow$

different reference points



M

$B=\{p \mid |z(p)|<\rho\}$ $\quad \overline{B}\subset$ coordinate neighborhood of q

$\mathcal{F}_c = \left\{ \begin{array}{l} v = \text{subharmonic on } M\setminus\overline{B},\quad v\le 1, \\ v\equiv 0 \text{ on } M\setminus\{\text{compact set} \supset \overline{B}\} \end{array}\right\}$: Perron family

$\Rightarrow \omega(p) = \sup\{v(p) \mid v\in\mathcal{F}_c\}$

$0\le\omega\le 1$ on $M\setminus\overline{B}$

Strong max/min principle $\Rightarrow$ either $\omega\equiv 1$ or $0<\omega<1$

$\boxed{\text{HW}}$ barrier function argument $\Rightarrow \omega(p)\to 1$ as $p\to\partial B$

Lemma if $\exists\, g(p,q)$ for some $q\in B$, then $0<\omega<1$ on $M\setminus\overline{B}$

if $0<\omega<1$ on $M\setminus\overline{B}$, the $g(p,q)$ exists $\forall q\in B$

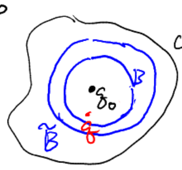
pf: 1° let $c=\min\limits_{|p|=\rho} g(p,q) > 0$

$\forall u\in\mathcal{F}_c$, max principle $\Rightarrow u\le \frac{g}{c}$ on $M\setminus\overline{B}$

$\Rightarrow \omega\le\frac{g}{c}$ on $M\setminus\overline{B}$

if $\inf\limits_{M\setminus\overline{B}}\omega>0 \Rightarrow \inf\limits_{M\setminus\{q\}} g>0 \rightarrow\leftarrow$

$\therefore \inf\limits_{M\setminus\overline{B}}\omega=0. \Rightarrow 0<\omega<1$ on $M\setminus\overline{B}$.

2°  coordinate neighborhood

(singularity $\log|z(p)-z(q)|$)

$u\in\mathcal{F}_q \Rightarrow u|_{\partial\tilde{B}} \le$ uniform bound

$M=\max\limits_{\partial\tilde{B}} u$

max principle
- $u(p)+\log|z(p)-z(q)|$ on $\tilde{B}$
- $u(p)$ on $M\setminus\overline{B}$

$u(p)+\log|z(p)-z(q)|$ is subharmonic

$\Rightarrow u(p)+\log|z(p)-z(q)| \le M + C$ $\quad C = \max\limits_{\partial B}\log|z(p)-z(q)|,\ \max\limits_{\partial\tilde{B}}\log|z(p)-z(q)|$

$\forall p\in\tilde{B}$

$\Rightarrow u(p)\le M+2C \quad \forall p\in\partial B$

$u-(M+2C)\,\omega$: subharmonic on $M\setminus\overline{B}$, $\limsup\limits_{p\to\partial B} = 0$

max principle $\Rightarrow u\le (M+2C)\,\omega$

on $\partial\tilde{B}$ $\quad M\le (M+2C)\,\kappa \rightarrow \max\limits_{\partial\tilde{B}}\omega\in(0,1)$

$\Rightarrow M\le \frac{2C\kappa}{1-\kappa}$ #

Cor. if $G(p,q)$ exists for some $q \Rightarrow$ exists for all q

Symmetry of Green's function

prop if Green's function exists, then $g(p,q)=g(q,p)$ $\forall p\ne q$

e.g. 1) on the unit disk D. $g(z,0)=-\log|z|$

by conformal change $g(z,w)=-\log\left|\frac{z-w}{1-\bar{w}z}\right|$

2) on $\mathbb{C}$, $g(z,0)$ does NOT exist

$g(z,0)=-\log|z| + ???$: harmonic & positive on $\mathbb{C}\setminus\{0\}$ impossible!

intuitive argument for symmetry:

Green's second identity $\qquad$ n: unit outer normal

$$\iint_\Omega v\Delta u - u\Delta v = \int_{\partial\Omega}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)ds$$

f: C^∞, compact support. $\Rightarrow$ $\iint_\Omega f(p)\,\Delta g(p,q) = -2\pi f(q)$

$$\iint g(p,q')(\Delta_p g(p,q)) \text{ "=" } -2\pi\, g(q,q')$$

$$\text{"="} \iint (\Delta_p g(p,q'))\, g(p,q) \text{ "=" } -2\pi\, g(q',q)$$

for the proof, work with the by-parts formula on g carefully.

bipolar Green's function $\qquad$ [Gamelin ; § XVI.5]

Not every Riemann surface admits a Green's function

However, it is fine if we allow two singularities.

e.g. on $\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$, $\quad G(z) = -\log|z|$

singularity at $z=0$ & $z=\infty$

defn $\quad q_1, q_2 \in M$, Δ_1, Δ_2: two disjoint coordinate neighborhood of q_1 & q_2

a bipolar Green's function with poles at q_1 and q_2

is a harmonic function $G(p,q_1,q_2)$ on $M\setminus\{q_1,q_2\}$

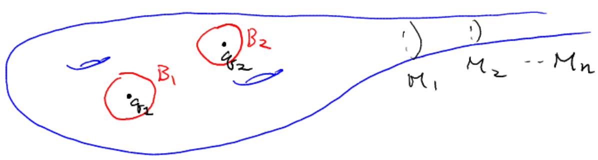
such that
$$\begin{cases} G(p,q_1,q_2) + \log|z_1(p)| \text{ is harmonic on } \Delta_1 \\ G(p,q_1,q_2) - \log|z_2(p)| \text{ is harmonic on } \Delta_2 \\ G(p,q_1,q_2) \text{ is bounded on } M\setminus(\Delta_1\cup\Delta_2) \end{cases}$$

rmk $\quad$ not unique, up to adding a bounded harmonic function on M

prop $\quad$ if the Green's function exists, the bipolar Green's function exists.

pf: $\quad$ take $G(p,q_1,q_2) = g(p,q_1) - g(p,q_2)$ $\quad$ #

$\hookrightarrow$ motivates the following idea



$\overline{M}_n$: compact exhaustion

construct $g_n(p,q_1)$, $g_n(p,q_2)$ on M_n

$$g_n(p,q_1) - g_n(p,q_2) \xrightarrow{n\to\infty} G(p,q_1,q_2)$$

existence of $g_n(p,q_1)$ on M_n

$\overline{M}_n \subset M$ (compact) $\qquad \partial M_n$ = finitely many analytic arcs

recall $\quad g_n(p,q_1)$ exists if and only if $w(p)$ is strictly between 0 & 1 where $w(p)$ is the upper envelope of

$\mathcal{F} = \{ u : \text{subharmonic on } M_n \setminus \overline{B_1} \mid u \le 1,\ u = 0 \text{ outside a compact subset of } M_n \}$

Hence, if $w(p) \xrightarrow{p \to \zeta_0 \in \partial U_\varepsilon} 0$, then $g_n(p, q_1)$ exists.

pf: $\mathcal{F} \subset \mathcal{F}_+ = \{ u : \text{subharmonic on } M_n \setminus \overline{B_1} \mid \limsup_{p \to \zeta_0 \in \partial M_n} u \le 0 \text{ and } \limsup_{p \to \zeta_0 \in \partial B_1} u \le 1 \}$

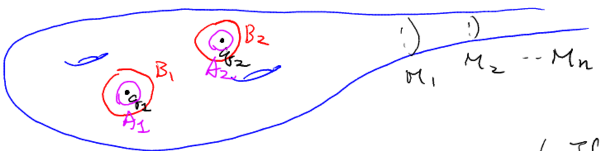
has compact closure in M

$\rightsquigarrow$ barrier function argument.

the upper envelope has boundary value $\begin{cases} 0 & \text{on } \partial M_n \\ 1 & \text{on } \partial B_1 \end{cases}$

thus, $\limsup_{p \to \zeta_0 \in \partial M_n} w \le 0$ ※

limits of bipolar Green's functions



say $A_j = \{ |z_j| < 6 < \rho \}$
$B_j = \{ |z_j| < \rho \}$

Fix ℓ. consider $\{ g_n(p, q_1) - g_n(p, q_2) \}_{n > \ell}$ on $p \in M_\ell \setminus (\overline{B_1} \cup \overline{B_2})$

(If it is uniformly bounded $\Rightarrow$ normal family of harmonic functions)

$C_1(n) = \max_{\partial A_1} g_n(p, q_1) \qquad C_2(n) = \max_{\partial A_2} g_n(p, q_2)$

max principle for $g_n(p, q_1) + \log |z_1(p)|$ on B_1

$\Rightarrow C_1 + \log 6 \le \max_{\partial B_1} g_n(p, q_1) + \log 6$

$\therefore \exists\ P_1 \in \partial B_1 \Rightarrow g_n(P_1, q_1) \ge C_1 - \log\left(\frac{6}{\rho}\right)$

$C_1 - g_n(P_1, q_1) \le \log\left(\frac{6}{\rho}\right)$

$\Rightarrow C_1 - g_n(p, q_1)$ is a positive harmonic function on $M_\ell \setminus \overline{A_1} \supset M_\ell \setminus (\overline{A_1} \cup \overline{A_2})$

(so as $C_2 - g_n(p, q_2)$) $\qquad g_n(p, q_1) = \sup\{ v(p) \mid v \in \mathcal{F}_{q} \}$ ...

By Harnack principle for $\partial B_1 \cup \partial B_2 \subset M_\ell \setminus (\overline{A_1} \cup \overline{A_2})$

$C_1 - g_n(p, q_1) \le \kappa (C_1 - g_n(P_1, q_1)) \le \kappa \log \frac{6}{\rho} \qquad \forall\, p \in \partial B_1 \cup \partial B_2$

$\Rightarrow C_1 - \kappa \log \frac{6}{\rho} \le g_n(p, q_1) \le C_1 \qquad C_1 = C_1(n), \quad \kappa$: NOT depend on n

But $g_n(p, q_1)$ is harmonic on B_2

$\Rightarrow C_1 - \kappa \log \frac{6}{\rho} \le g_n(q_2, q_1) \le C_1$

Similarly, $C_2 - \kappa \log \frac{6}{\rho} \le g_n(q_1, q_2) \le C_2$

Since $g_n(q_2, q_1) = g_n(q_1, q_2) \Rightarrow C_2 - \kappa \log \frac{6}{\rho} \le C_1$

$C_1 - \kappa \log \frac{6}{\rho} \le C_2$

$\Rightarrow |C_1 - C_2| \le \kappa \log \frac{6}{\rho}$: independent of n !

Then, $|g_n(p, q_1) - g_n(p, q_2)| \le 2\kappa \log \frac{6}{\rho} \qquad \forall\, p \in \partial B_1 \cup \partial B_2$

$\Rightarrow$ maximum principle, holds $\forall\, p \in M_\ell \setminus (\overline{B_1} \cup \overline{B_2})$

(on B_1 & B_2, consider $g_n(p, q_1) - g_n(p, q_2) + \log|z_1(p)| - \log|z_2(p)|$) $\#$

uniformization theorem

thm M: simply-connected Riemann surface
$\Rightarrow$ M is conformally equivalent to D, $\mathbb{C}$ or $\hat{\mathbb{C}}$

1° simply-connected & analytic continuation

simply-connected (condition in topology)

any two paths can deformed to each other, or any loop can shrink to a point

u: harmonic on M $\to$ construct a harmonic conjugate on any small neighborhood of M, unique up to constants

Fix its value at p $\to$ for any $q \in M \setminus \{p\}$ and any path from p to q, $\exists$ holomorphic function on the neighborhood of the path $\to$ real part $= u$

If we deform the path a little bit, the analytic function is the same on some small neighborhood of q

Simply-connectedness $\Rightarrow$ $\exists !$ analytic function whose real part is u, and has the prescribed value (of imaginary part) at p.

2° M: simply-connected, and M admits the Green's function

at first $\exists$ φ: holomorphic on a neighborhood of q_0

$\to |\varphi(p)| = e^{-g(p, q_0)}$ (q_0 is a simple zero of φ)

apply the above analytic continuation argument to φ ← globally defined

(on $M \setminus \{p\}$, can construct harmonic conjugate of $g(p, q_0)$ locally $\rightsquigarrow h(p) \rightsquigarrow \varphi = e^{-(g + ih)}$)

Since $g > 0$ and $|\varphi| = e^{-g}$, $|\varphi| < 1$ on M

Since g blows up at q_0 only, φ has only one zero at q_0

Is φ injective? For any $q_1 \in M \setminus \{q_0\}$

Consider $\psi(p) = \dfrac{\varphi(p) - \varphi(q_1)}{1 - \overline{\varphi(q_1)}\,\varphi(p)}$ $\quad \begin{cases} \psi(q_1) = 0 \\ \psi(q_0) = -\varphi(q_1) \end{cases}$

If $u \in \mathcal{F}_{q_1}$, $u(p) + \frac{1}{n}\log|\psi(p)|$ is subharmonic on M $\quad$ n: vanishing order of ψ at q_1

Since $u(p) + \frac{1}{n}\log|\psi(p)| < 0$ outside a compact set

$\Rightarrow u(p) + \frac{1}{n}\log|\psi(p)| < 0$ everywhere

$\Rightarrow g(p, q_1) + \frac{1}{n}\log|\psi(p)| \leq 0 \quad \forall p \in M$

$$\Rightarrow \quad g(q_0,q_1) + \tfrac{1}{n}\log|\psi(q_0)| = g(q_0,q_1) - \tfrac{1}{n}\, g(q_1,q_0)$$

$$\underset{\text{sym of } g}{=} \frac{n-1}{n}\, \underbrace{g(q_1,q_0)}_{\text{positive}} \le 0$$

By the strong max principle, $n=1$, and $g(p,q_1) = -\log|\psi(p)|$

It follows that the only zero of ψ is q_1

$\Rightarrow$ $\varphi: M \to D$ injective, harmonic

$\varphi(M)$: simply-connected , By Riemann mapping $\Rightarrow$ DONE

3° an existence criterion for Green's function

lemma if $\exists$ bounded, non-constant holomorphic function on M then the Green's function exists.

pf: Assume $f(q)=0$. $|f(p)|<1 \quad \forall p \in M$

Similar as above, $u(p) + \frac{1}{n}\log|f(p)| < 0 \quad \forall p \in M,\ u \in \mathcal{F}_q$

※

4° Green's function does NOT exist

$G(p, q_1, q_2)$: bipolar Green's function

Simply-connected $\Rightarrow$ $\exists\ \varphi(p)$ = holomorphic $\to$ $|\varphi(p)| = e^{-G(p,q_1,q_2)}$

φ has a simple zero at q_1, a simple pole at q_2

and $\frac{1}{C} \le |\varphi(p)| \le C \quad \forall p \in M\setminus(B_1 \cup B_2)$ pole at q_2

Is φ injective? $q_0 \in M\setminus\{q_1,q_2\} \rightsquigarrow \varphi(p)-\varphi(q_0)$?

Consider $\tilde\varphi$: $|\tilde\varphi(p)| = e^{-G(p,q_0,q_2)}$

pole at q_2

and $\psi(p) = \dfrac{\varphi(p)-\varphi(q_0)}{\tilde\varphi(p)}$: no poles

$\Rightarrow$ ψ is a bounded holomorphic function on M

Since the Green's function does NOT exist, ψ = constant

$\Rightarrow$ $\psi(p) = \psi(q_1) \ne 0$

$\Rightarrow$ $\varphi(p) = \varphi(q_0)$ only for $p = q_0$

$\Rightarrow$ $\varphi: M \to \hat{\mathbb{C}}$, injective. image is simply-connected

By the Riemann mapping theorem. if $\varphi(M) \subsetneq \hat{\mathbb{C}}\setminus\{\text{points}\}$

$\Rightarrow M \cong \varphi(M) \cong D$

$\exists$ Green's function $\to\leftarrow$

Hence $\varphi(M) = \hat{\mathbb{C}}$ or $\hat{\mathbb{C}}\setminus\{\text{point}\} \cong \mathbb{C}$. ※

For general M.

topology $\rightsquigarrow$ $\pi_1(M)$: fundamental group. which measures the failure of shrinking loops to a point

$\exists\, \tilde{M} \xrightarrow{\pi} M$ $\quad$ $\tilde{M}$: simply-connected $\quad$ $M = \tilde{M} / \pi_1(M)$

$\pi_1(M)$ acts on $\tilde{M}$: fixed point free & properly discontinuous

If M is a Riemann surface, so is $\tilde{M}$

and $\pi_1(M) \subset \mathrm{Aut}(\tilde{M})$

$\mathbb{C}$ $\quad$ $\mathrm{Aut}(\mathbb{C}) = \{az+b\}$

$\hat{\mathbb{C}}$ $\quad$ $\mathrm{Aut}(\hat{\mathbb{C}}) = PSL_2(\mathbb{C})$ $\quad$ $\longrightarrow$ cannot produce $M_{g>1}$

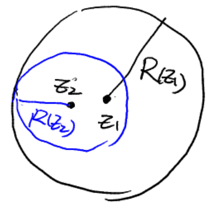
D $\quad$ $\mathrm{Aut}(D) = \left\{ e^{i\theta} \frac{z-a}{1-\bar{a}z} \;\middle|\; |a|<1 \right\}$

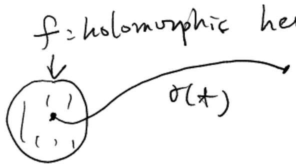
appendix (analytic continuation)

- $f(z)$: holomorphic on $D \subset \mathbb{C}$

  $R(z)$: radius of convergence at z

  $\Rightarrow$ $|R(z_1) - R(z_2)| \leq |z_1 - z_2|$



- 

  f is analytically continuable along γ

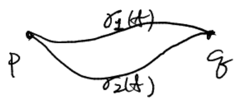
  if $\exists\, f_t(z) = \sum_{n=0}^{\infty} a_n(t)\,(z-\gamma(t))^n$ $\quad$ $|z-\gamma(t)| < R(t)$

  such that $f_s(z) = f_t(z)$ if $|z-\gamma(s)| < R(s)$, $|z-\gamma(t)| < R(t)$

  if $\exists$

  $\Rightarrow$ $a_n(t)$ & $R(t)$ depends continuously in t

  and it is unique

- 

  if $\gamma_1(t)$ & $\gamma_2(t)$ are closed to each other

  $\Rightarrow$ obtain the same function on a neighborhood of q (same germs)

  hence, homotopy paths $\Rightarrow$ same result.