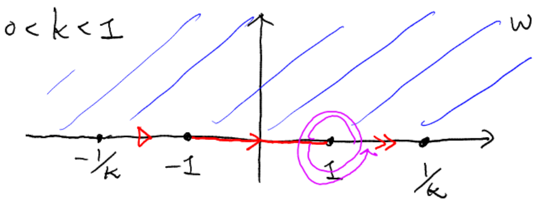
Recap.

1° recall: Schwarz – Christoffel formula [Ahlfors, § 2.2 of ch. 6]

conformal map between $\mathbb{H} \cong D$ and polygon

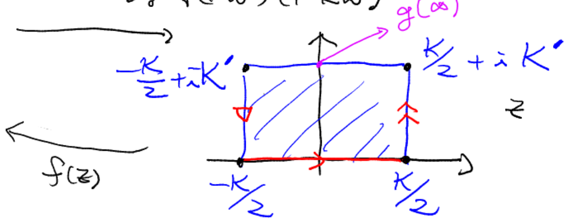
focus on the following case:

$0 < k < 1$



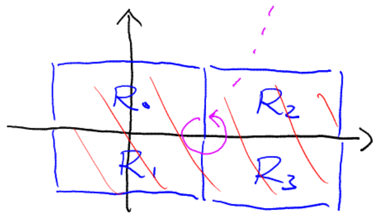
$$g(w) = \int_0^w \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}$$: elliptic integral

bad notation



2° Denote the inverse map by $f(z)$

By the Schwarz reflection principle. $f(z)$ can be extended to a meromorphic function on $\mathbb{C}$



e.g. $z \in R_1$, $f(z) := \overline{f(\bar{z})}$

$z \in R_2$, $f(z) = \overline{f(K - \bar{z})}$

$\Rightarrow$ It is not hard to check that the extension is doubly periodic with periods $2K$ & $2iK'$

Use this fundamental domain

Namely. $f(z + 2K) = f(z) = f(z + 2iK')$

3° Hence. $f : M = \mathbb{C}/2K + 2iK' \longrightarrow \hat{\mathbb{C}}$ a holomorphic map from a genus one Riemann surface to the Riemann sphere.

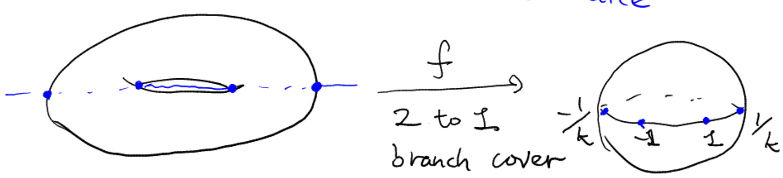
degree & branch points?

For instance, $f^{-1}(i) = \{$two points$\}$ one in R_0. another in R_3

By the Riemann – Hurwitz formula. (midterm #5)

there are 4 branch points on M. (with distinct images in $\mathbb{C}$)

By construction. $\{ \frac{K}{2} + \underbrace{nK + mK'}_{\text{"half" lattice}} \}$ are the branch points.



4° elliptic integral ~ cannot evaluate

Abel & Jacobi: elliptic integral is the inverse of an elliptic function (doubly periodic meromorphic function)

(NOT from conformal mapping & Schwarz – Christoffel formula)

Jacobi inversion: $g(w)$ is surjective to $M = \mathbb{C}/_{2K+2iK'}$

$$z = \int_0^{f(z)} \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}$$ ← in a way, we are evaluating $\int_0^z dz$ by using a pretty bad coordinate system.

basis for $\mathcal{H} \cong \mathbb{C}$

5° $\frac{d}{dz} \Rightarrow 1 = \frac{1}{\sqrt{(1-f^2)(1-k^2f^2)}} f' \Leftrightarrow (f')^2 = (1-f^2)(1-k^2f^2)$

$f, f' \in \mathcal{M}(M)$ In fact, $M \to \mathbb{CP}^2 = \mathbb{C}^3\setminus\{0\}/\sim$

$z \longmapsto [1 : f : f']$

is a holomorphic embedding to

$\frac{\zeta_1}{\zeta_0}$ $\frac{\zeta_2}{\zeta_0}$

$(\zeta_0, \zeta_1, \zeta_2) \sim \lambda(\zeta_0, \zeta_1, \zeta_2)$ $\lambda \in \mathbb{C}\setminus\{0\}$

$$\left(\frac{\zeta_2}{\zeta_0}\right)^2 = \left(1-\left(\frac{\zeta_1}{\zeta_0}\right)^2\right)\left(1-k^2\left(\frac{\zeta_1}{\zeta_0}\right)^2\right)$$

$$\Leftrightarrow \zeta_0^2\zeta_2^2 = (\zeta_0^2 - \zeta_1^2)(\zeta_0^2 - k^2\zeta_1^2)$$

6° another direction: start with $M = \mathbb{C}/L$ $L = \mathbb{Z}\langle\omega_1, \omega_2\rangle$

linearly independent over $\mathbb{R}$

[Ahlfors, §3 of ch. 7]

let $\wp(z) = \frac{1}{z^2} + \sum_{L\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$ (Weierstrass $\wp$-function)

$\Rightarrow \wp(z) = \wp(z+\omega_1) = \wp(z+\omega_2)$: doubly periodic

$\Rightarrow \wp(z) \in \mathcal{M}(M)$ or $M \xrightarrow{\wp} \hat{\mathbb{C}}$ of degree 2

only one double pole

relation to elliptic integral?

$\wp'(z)$ is also a meromorphic function on M.

both $\wp(z)$ & $\wp'(z)$ only has one poles.

work on a local chart for $[0] \in M$

$\Rightarrow \wp(z) = \frac{1}{z^2} + 3G_2 z^2 + 5G_3 z^4 + \cdots$

some coefficient

$\wp'(z) = -\frac{2}{z^3} + 6G_2 z + 20 G_3 z^3 + \cdots$

As in 5°, try to cancel the singular part of $(\wp'(z))^2$

$(\wp'(z))^2 = \frac{4}{z^6} - \frac{24G_2}{z^2} - 80G_3 + \cdots$

$4(\wp(z))^3 = \frac{4}{z^6} + \frac{36G_2}{z^2} + 60G_3 + \cdots$

$60G_2\wp(z) = \frac{60G_2}{z^2} + \cdots$

$\Rightarrow (\wp'(z))^2 - 4(\wp(z))^3 - 60G_2\wp(z) = -140G_3 + \cdots$

double periodic, possible pole at lattice L.

but regular at $z=0$

$\Rightarrow$ constant function : $M \to \hat{\mathbb{C}}$, $-140G_3$

Rewrite it as. $(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3$

$$\Rightarrow z - z_0 = \int_{\wp(z_0)}^{\wp(z)} \frac{dw}{\sqrt{4w^3 - g_2 w - g_3}}$$