Abel–Jacobi Theorems [FK ; § III.6]

$g=0$ $\quad M \cong \hat{\mathbb{C}}$. $\quad$ a divisor $\mathfrak{A}$ with $\deg \mathfrak{A} = 0$ $\quad$ (say, $\mathfrak{A} \neq 1$)

$\Leftrightarrow$ $\mathfrak{A} = (f)$ for some $f \in \mathcal{M}^*$

($\Rightarrow$ construct the rational function by hand)

Q $g \geq 1$ $\quad$ if $\deg \mathfrak{A} = 0$. is $\mathfrak{A}$ principal ?

the Jacobian variety and the Abel–Jacobi embedding

(compare with the previous digression on $\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$)

(period integral $\longleftrightarrow$ holomorphic / meromorphic functions, complex structures)

Start with a $2g$-loops-cut $\{A_1, \cdots, A_g, B_1, \cdots, B_g\}$

Let $\{\zeta_1, \cdots, \zeta_g\}$ be the basis of $\mathcal{H}$ such that $\int_{A_j} \zeta_k = \delta_{jk}$

Riemann $\Rightarrow$ $[\int_{B_j} \zeta_k]$ has symmetric, positive definite imaginary part

Fix $P_0 \in M$. for any $P \in M$

consider $\varphi : M \to \mathbb{C}^g / ?$

$P \longmapsto \left(\int_{P_0}^{P} \zeta_1, \cdots, \int_{P_0}^{P} \zeta_g\right)$ $\quad$ any path from P_0 to P

Ambiguity : for different choice of paths.

$\tilde{\varphi}(P) - \varphi(P) = \left(\int_\gamma \zeta_1, \cdots, \int_\gamma \zeta_g\right)$ $\quad$ γ : closed loop at P

Since $H_1(M) \cong \mathbb{Z}\langle A_1, \cdots, A_g, B_1, \cdots, B_g\rangle$, $\quad [\gamma] = \sum_{j=1}^{g} m_j [A_j] + n_j [B_j]$

$$\Rightarrow \tilde{\varphi}(P) - \varphi(P) = \sum_{j=1}^{g} \left(m_j \left(\int_{A_j}\zeta_1, \cdots, \int_{A_j}\zeta_g\right) + n_j \left(\int_{B_j}\zeta_1, \cdots, \int_{B_j}\zeta_g\right)\right)$$

$$= \sum_{j=1}^{g} \left(m_j e^{(j)} + n_j \pi^{(j)}\right)$$

$$\begin{array}{cc} & \begin{array}{ccc} \int_{A_j} & & \int_{B_j} \end{array} \\ \zeta_k & \left[\begin{array}{c:c} I & \Pi \end{array}\right] \end{array}$$

column vectors

Since $\operatorname{Im} \Pi$ is invertible (symmetric & positive definite)

$\{e^{(j)}, \pi^{(j)}\} \in \mathbb{C}^g$ are linearly independent over $\mathbb{R}$

<u>defn</u> $J(M) = \mathbb{C}^g / \mathbb{Z}\langle e^{(1)}, \cdots, e^{(g)}, \pi^{(1)}, \cdots, \pi^{(g)}\rangle$ is called the Jacobian variety of M

$\varphi : M \to J(M)$ is called the Abel–Jacobi map

<u>about $J(M)$</u> : a g-dimensional complex manifold.

(in fact, $\exists$ charts whose transitions are translation)

a Lie group, $+ : M \times M \to M$ & inverse are smooth maps

Prop $\varphi: M \to J(M)$ is holomorphic and has maximal rank (immersion)

Pf:

z: coordinate near P w/ $z(P)=0$

$\zeta_j = h_j'(z)\,dz \qquad h_j(z)$: holomorphic

$$\varphi(z) = \varphi(0) + \left(\int_0^z \zeta_1, \cdots, \int_0^z \zeta_g\right) = \varphi(0) + (h_1(z), \cdots, h_g(z))$$

(z: points near P; 0: P)

$\Rightarrow \varphi$ is holomorphic near P

$\mathbb{C}^g$ locally $\cong \varphi(U)\times\mathbb{C}^{g-1}$ $\overset{\text{implicit function theorem}}{\Longleftrightarrow}$ (maximal rank $\varphi: \underset{Q}{\mathcal{U}} \subset \mathbb{C}^1 \to \mathbb{C}^g$ in coordinate, $d\varphi^{(j)}|_Q \neq 0$ for some j)

$$d\varphi = (\zeta_1, \cdots, \zeta_g)$$

Since $i(1) - i(Q) \underset{\boxed{HW}}{=} 1$, $\exists\, \zeta \in \mathcal{H},\ \zeta|_Q \neq 0 \Rightarrow$ maximal rank #

injectivity of the Abel-Jacobi map

P, Q if $\varphi(P) = \varphi(Q) \in \mathbb{C}^g / \underset{\text{lattice}}{L(M)} = \mathbb{Z}\langle e^{(1)}, \cdots, e^{(g)}, \pi^{(1)}, \cdots, \pi^{(g)}\rangle$

$$\Rightarrow \left(\int_Q^P \zeta_1, \cdots, \int_Q^P \zeta_g\right) = \sum_{j=1}^g \left(m_j e^{(j)} + n_j \pi^{(j)}\right)$$

recall [bilinear relation $\displaystyle\int_Q^P \zeta_j = \frac{1}{2\pi i}\int_{B_k} \tau_{PQ}$

P, Q: not on the loops-cuts

τ: holomorphic on $M \setminus \{P, Q\}$ $\qquad \displaystyle\int_{A_j} \tau = 0$

$\operatorname{ord}_P \tau = -1 = \operatorname{ord}_Q \tau$. $\operatorname{Res}_P \tau = 1 = -\operatorname{Res}_Q \tau$]

$$\Rightarrow \int_{A_j} \tau = 0, \qquad \int_{B_k} \tau = 2\pi i\left(m_k + \sum_{j=1}^g n_j \int_{B_j} \zeta_k\right)$$

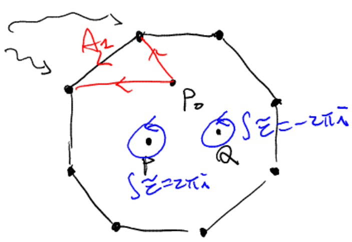
Consider $\displaystyle\tilde{\tau} = \tau + \sum_{j=1}^g c_j \zeta_j$

$$\int_{A_k} \tilde{\tau} = c_k, \qquad \int_{B_k} \tilde{\tau} = 2\pi i\left(m_k + \sum_{j=1}^g n_j \pi_{kj}\right) + \sum_{j=1}^g c_j \underset{=\pi_{kj}}{\pi_{jk}}$$

Take $c_j = -2\pi i n_j \Rightarrow \displaystyle\int_{A_k} \tilde{\tau} = -2\pi i n_k$, $\displaystyle\int_{B_k} \tilde{\tau} = 2\pi i m_k$ (in $2\pi\mathbb{Z}$)

We can consider $\exp\left(\int \tilde{\tau}\right)$ on M

differ by $\times \exp\left(\int_{A_1}\tilde{\tau}\right) = \exp(2\pi i \mathbb{Z}) = 1$



$$f(R) = \exp\left(\int_{P_0}^R \tilde{\tau}\right)$$

$\Rightarrow f$ is a well-defined meromorphic function on M with poles at P & Q

near P $\quad \dfrac{df}{f} = \dfrac{1}{z} + (\text{regular part})$ $\quad \overset{\text{argument principle}}{\Longrightarrow}$ $\operatorname{ord}_P f = 1$

near Q $\quad \dfrac{df}{f} = \dfrac{-1}{w} + (\text{regular part})$ $\quad \operatorname{ord}_Q f = -1$

$\Rightarrow f$ has only one simple pole $\Rightarrow f: M \xrightarrow{\sim} \hat{\mathbb{C}}$

contradiction #

Thm $\varphi: M \longrightarrow \underset{\mathbb{C}^g/L(M)}{\underset{\shortparallel}{J(M)}}$ is injective, and thus a (holomorphic) embedding. #

Cor When $g=1$, $\varphi: M \to \mathbb{C}^1/L(M)$ is a biholomorphism

topologically a torus with a Riemann surface structure

NOT only a Riemann surface but comes with an abelian group structure

almost gives a group structure (up to P_0. the "zero")

Principal divisors

thm (Abel) $\mathfrak{A} \in \mathrm{Div}^{(0)}(M) = \{\text{degree zero divisors}\}$, $\mathfrak{A} \neq 1$

$\varphi(\mathfrak{A}) = 0 \iff \mathfrak{A}$ is principal, $\mathfrak{A} = (f)$ $f \in \mathcal{M}^*(M)$

pf: 0° $\mathfrak{A} = P_1^{\alpha_1} \cdots P_t^{\alpha_t} / Q_1^{\beta_1} \cdots Q_s^{\beta_s}$ $\quad \begin{cases} P_i \neq Q_j & \forall i,j \\ P_i \neq P_j,\ Q_i \neq Q_j & \forall i \neq j \end{cases}$

$\alpha_i > 0$, $\beta_j > 0$, $\sum_{i=1}^{t} \alpha_i = \sum_{j=1}^{s} \beta_j$

$$\varphi(\mathfrak{A}) = \sum_{i=1}^{t} \alpha_i \varphi(P_i) + \sum_{j=1}^{s} \beta_j \varphi(Q_j)$$

$\boxed{\text{check}}$ independent of the choice of P_0

1° $\Leftarrow$) Suppose that $\mathfrak{A} = (f)$

From the above discussion, consider $\frac{df}{f}$

(choose the loops-cuts avoiding P_0 & $\mathfrak{A}$)

$$\Rightarrow \frac{df}{f} - \sum_{i=1}^{t} \alpha_i \tau_{P_i} + \sum_{j=1}^{s} \beta_j \tau_{Q_j} \in \mathcal{H}$$

$\mathrm{ord}_{P_i} = -1 = \mathrm{ord}_{P_0}$ and $\int_{A_k} = 0$

$\mathrm{Res}_{P_i} = 1 = -\mathrm{Res}_{P_0}$

$$\frac{df}{f} = \sum_{i=1}^{t} \alpha_i \tau_{P_i} - \sum_{j=1}^{s} \beta_j \tau_{Q_j} + \sum_{k=1}^{g} c_k \zeta_k \qquad c_i \in \mathbb{C}$$

$$\Rightarrow \begin{cases} \int_{A_l} \frac{df}{f} = c_l \\ \int_{B_l} \frac{df}{f} = 2\pi i \left(\sum_{i=1}^{t} \alpha_i \int_{P_0}^{P_i} \zeta_l - \sum_{j=1}^{s} \beta_j \int_{P_0}^{Q_j} \zeta_l \right) + \sum_{k=1}^{g} c_k \pi_{kl} \end{cases}$$

But $\frac{df}{f} = d \log f \xrightarrow[\substack{\text{closed} \\ \text{curve} \\ f \neq 0}]{\int} 2\pi i \mathbb{Z}$

$$\Rightarrow \left(\sum_{i=1}^{t} \alpha_i \int_{P_0}^{P_i} \zeta_l - \sum_{j=1}^{s} \beta_j \int_{P_0}^{Q_j} \zeta_l \right)_{l=1}^{g} \in \mathbb{Z} \langle e^{(1)}, \ldots, e^{(g)}, \pi^{(1)}, \ldots, \pi^{(g)} \rangle$$

2° $\Rightarrow$) similar as the proof for $\varphi(P) = \varphi(Q) \Rightarrow P = Q$ #

$\Updownarrow$

$\varphi(PQ^{-1}) = 0$

Surjectivity of φ?

$\varphi: M \to J(M) = \mathbb{C}^g / L(M)$ is a holomorphic embedding

For $D = P_1 \cdots P_n \xrightarrow{\varphi} \sum_{j=1}^{n} \varphi(P_j) \in J(M)$

defn $M_n = \{ D \in \mathrm{Div}(M) \mid \deg D = n,\ D > 1 \}$

Consider $\varphi : M_n \to J(M)$

Minkowski addition

(roughly speaking , $\varphi(M_n) = \varphi(M) + \cdots + \varphi(M)$)

Q For $n >> 1$, will $\varphi : M_n \to J(M)$ surjective ?

(shall at least have g freedoms. look at M_g)

Special divisors

defn $D \in M_g$. D is special if $i(D) > 0 \overset{R\text{-}R}{\iff} r(D) = g + (1-g) + i(D) > 1$

$\Omega(D)$: g-conditions on $\Omega(1) \cong \mathbb{C}^g$

D : special $\iff$ linear dependent conditions

prop $\forall D \in M_g$, $\exists D' \in M_g$ close to D such that D' is NOT special. Moreover, D' can be chosen to consist of distinct g points

pf: $D = P_1 \cdots P_n$, for each P_j, chose an open neighborhood U_j

goal $D' = P_1' \cdots P_n'$ $\quad P_j' \in U_j$ ($\iff D'$ close to D)

$D_1' = P_1'$ $\quad D_j' = D_{j-1}' P_j'$ $\quad i(D_j') = g - j$ $\quad$ ($\geq$ always)

$j = 1$. Since $i(Q) = g-1$ $\forall Q \in M$, take $P_1' = P_1$

induction. $i(D_j') = g - j$ $\quad \{\varphi_1, \cdots, \varphi_{g-j}\}$: basis for $\Omega(D_j')$

(holomorphic on U_{j+1}

if $\varphi_1(P_{j+1}) \neq 0$, take $P_{j+1}' = P_j$

if $\varphi_1(P_{j+1}) = 0$, $\varphi_1(P_{j+1}') \neq 0$ $\forall P_{j+1}' \neq P_{j+1}$ near P_{j+1} #

Jacobi inversion

thm $\varphi : M_g \to J(M)$ is surjective.

Namely, any point in $J(M)$ is the image of a divisor > 1 of degree g.

pf: (local) let $D_0 \in M_g$: not special, $D_0 = P_1 \cdots P_g$: g distinct points

(U_j, z_j) : coordinate on a neighborhood of P_j, $z_j(P_j) = 0$

Consider φ on $U_1 \times U_2 \times \cdots \times U_g$

$$(z_1, z_2, \cdots, z_g) \longmapsto \underset{\sum_{j=1}^{g} \varphi(P_j)}{\underset{\shortparallel}{\varphi(D_0)}} + \sum_{j=1}^{g} \left(\int_0^{z_j} \zeta_1 \cdots , \int_0^{z_j} \zeta_g \right)$$

$\Rightarrow d_{z_j}\varphi = (\zeta_1(z_j), \cdots, \zeta_g(z_j))$

d in z_j-direction

column vector

or $d\varphi = \begin{bmatrix} \zeta_1(z_1) & \cdots & \zeta_1(z_g) \\ \vdots & & \vdots \\ \zeta_g(z_1) & \cdots & \zeta_g(z_g) \end{bmatrix}$

at $(z_1, \cdots, z_g)$

$\Uparrow$

$U_1 \times \cdots \times U_g$

$\mathbb{C}^g \to \mathbb{C}^g$

$i(D_0) = 0$

$\iff (d\varphi)^*\big|_{(0,\cdots,0)}$ is invertible

By inverse function theorem (on $\varphi : U_1 \times \cdots \times U_g \subset \mathbb{C}^g \to \mathbb{C}^g$)

$\exists$ neighborhood V of $\varphi(D_0)$ in $J(M)$ (or $\mathbb{C}^g$)

such that $V \subset \varphi(\mathcal{U}_1 \times \cdots \times \mathcal{U}_g) \subset \varphi(M_g)$

(global) : $\forall\, c \in \mathbb{C}^g$. $\exists\, N \in \mathbb{N}$

such that $\varphi(D_0) + \frac{c}{N} \in V$

i.e. $c = N(\varphi(Q_1 \cdots Q_g) - \varphi(D_0))$ for some $Q_j \in \mathcal{U}_j$

$\overset{?}{=} \varphi(D)$ for some $D \in M_g$

but $\varphi(P_0) = 0$ $\to$ deg $= g$

$\Rightarrow c = \varphi(\underbrace{(Q_1 \cdots Q_g)^N D_0^{-N} P_0^g}_{\deg = g})$

<u>recall</u> $r(\mathcal{A}^{-1}) > 0 \iff \mathcal{A} \sim \mathcal{B} \geq 1$

$r(\mathcal{A}^{-1}) = g + 1 - g + i(\mathcal{A}) > 0$

in fact deg $= g$ is always equivalent to an effective one

$\Rightarrow \frac{(Q_1 \cdots Q_g)^N}{D_0^N} P_0^g \sim \mathcal{B} \geq 1$ $\quad$ deg $\mathcal{B} = g$

By Abel. $\varphi(\mathcal{B}) = c \pmod{L(M)}$