Weierstrass Points [FK; § III.5]

$r(P^{-n}) = n+1-g+i(P^n) \geq n+1-g$

$f \in L(P^{-n}) \iff f: M \to \hat{\mathbb{C}},\ \deg f \leq n \quad f^{-1}(\infty) = \{P\}$

$f \in L(P^{-n}) \setminus L(P^{-(n-1)}) \iff \deg f = n$

<u>recall</u> $\deg K = 2g-2 \qquad i(P^n) = r(P^n K^{-1}) = 0$ when $n > 2g-2$, $n \geq 2g-1$

$\deg = n-(2g-2)$

$\Rightarrow$ When $n \geq 2g-1$, $r(P^{-n}) = n+1-g$

n	0	1	$\cdots$	$2g-2$	$2g-1$	$2g$
$r(P^{-n})$	1	$\cdots$	$\cdots$	?	g	$g+1$

non-decreasing stable

<u>Goal</u> ($g \geq 2$) except for finitely many P

n	0	1	$\cdots$	g	$g+1$	$\cdots$	$2g-2$	$2g-1$	$2g$
$r(P^{-n})$	1	1	$\cdots$	1	2		$g-1$	g	$g+1$

i.e. for finitely many P

$\exists f: M \to \hat{\mathbb{C}},\ \deg f \leq g \quad f^{-1}(\infty) = \{P\}$

<u>rmk</u> intuitively, jumps earlier is a closed condition $\Rightarrow$ discrete points.

※ These points are called the Weierstrass points of M.

They carry the information of the Riemann surface structure of M

(for instance, $\Rightarrow$ finite automorphism)

Weierstrass gap theorem (assume $g \geq 1$)

<u>thm</u> $\exists$ g positive integers, $1 = n_1 < n_2 < \cdots < n_g < 2g$

$\Rightarrow r(P^{-n_k}) = r(P^{-(n_k-1)})$

$\overset{R-R}{\iff} i(P^{n_k}) = i(P^{n_k-1}) + 1 \iff \nexists f:$ holomorphic on $M \setminus \{P\}$, $\mathrm{ord}_P f = -n_k$

<u>rmk</u> i) these n_k are called <u>gaps</u> at P (where $r(P^*)$ does NOT jump)

non-gaps $= \mathbb{N} \setminus \{\text{gaps}\}$.

$\boxed{\text{HW}}$ if i, j: non-gaps $\Rightarrow$ $i+j$: also non-gap

ii) For most P's, $n_1 = 1, \cdots, n_g = g$

pf: It is almost a direct consequence of the above discussion.

It remain to check that $r(P^{-k}) - r(P^{-(k-1)}) \leq 1$ & $n_1 = 1 \leftarrow \boxed{\text{HW}}$

$L(P^{-(k-1)}) \subset L(P^{-k})$ if codimension ≥ 2

$(f) \geq P^{-(k-1)} \quad (f) \geq P^{-k}$ $\quad \exists g_1, g_2 \in L(P^{-k}) \setminus L(P^{-(k-1)})$ & linearly indep.

$\Rightarrow \exists c \to g_1 - c g_2 \in L(P^{-(k-1)}) \to\leftarrow$ ※

some properties of gaps / non-gaps

let $1 < \alpha_1 < \cdots < \alpha_g = 2g$ be the first g <u>non-gaps</u>

$\exists f$: holomorphic on $M \setminus \{P\}$, $\mathrm{ord}_P f = -\alpha_k$

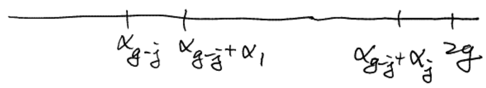
$1, 2, \cdots, 2g-1, 2g \qquad g$ gaps, g non-gaps

$\underset{1}{\Psi} \qquad \underset{2g}{\Psi}$

i) $\alpha_j + \alpha_{g-j} \geq 2g \qquad j \in \{1, 2, \cdots, g-1\}$

$\alpha_j + \alpha_{g-j}$ is also a non-gap.

if $\alpha_j + \alpha_{g-j} < 2g$. $\Rightarrow$ $\alpha_k + \alpha_{g-j} < 2g$, $\forall\, k \le j$



$\Rightarrow \exists\, j$ non-gaps $\in (\alpha_{g-j},\ 2g = \alpha_g)$

$\Rightarrow \exists\, (g+1)$ non-gaps in $[1, 2g]$ $\rightarrow\leftarrow$

dichotomy

ii) if $\alpha_1 = 2$, then $k\alpha_1$ is also a non-gap $\Rightarrow$ $\alpha_k = 2k$ $\quad k \in \{1, 2, \dots, g\}$

$\Rightarrow$ $\alpha_j + \alpha_{g-j} = 2g$

iii) if $\alpha_1 > 2$, then $\exists\, j \in \{1, \dots, g-1\}$ $\Rightarrow$ $\alpha_j + \alpha_{g-j} > 2g$ $\boxed{\text{HW}}$

iv) sum i) from $j=1$ to $g-1$ $\Rightarrow$ $2\sum_{j=1}^{g-1} \alpha_j \ge 2(g-1)g$

$\Rightarrow$ $\sum_{j=1}^{g-1} \alpha_j \ge g(g-1)$

by ii) & iii) "=" if and only if $\alpha_1 = 2$

Weierstrass points

$r(P^{-n})$: NOT look like this :

n	0	1	…	g	$g+1$	……	$2g-2$	$2g-1$	$2g$
$r(P^{-n})$	1	1	…	1	2		$g-1$	g	$g+1$
$i(P^n)$	g	$g-1$		0	0		0	0	0

defn P is called a Weierstrass point if $r(P^{-g}) \ge 2$ $\iff$ $i(P^g) > 0$

$r(P^{-g}) = \deg(P^g) + (1-g) + i(P^g)$

$\Rightarrow$ $i(P^g) = r(P^{-g}) - 1 > 0$ $\qquad r(P^{-n}) = n + 1 + g - i(P^n)$

Cor P is a Weierstrass point if and only if $\exists$ (nonzero) holomorphic differential ω with $\operatorname{ord}_P \omega \ge g$

Since $\mathcal{H} \cong \mathbb{C}^g$, the condition is easier to handle.

$\rightsquigarrow$ look for where $\mathcal{H}$ has "higher vanishing order"

Wronskian

$D \subset \mathbb{C}$: open & connected

A: a n-dim'l space of holomorphic functions on D

For any $z \in D$, we may choose a basis for A: $\{\varphi_1, \dots, \varphi_n\}$

such that $\operatorname{ord}_z \varphi_1 < \operatorname{ord}_z \varphi_2 < \dots < \operatorname{ord}_z \varphi_n$

($\mu = \min_{\varphi \in V} \{\operatorname{ord}_z \varphi\}$ choose φ_1 with $\operatorname{ord}_z \varphi_1 = \mu$.

let $A_1 = \{\varphi \in A \mid \operatorname{ord}_z \varphi > \mu\}$ $\iff$ z^μ-coefficient of $\varphi = 0$

A_1 has codimension 1 …)

Denote these vanishing orders by $\mu_1, \dots, \mu_n$

$$\begin{vmatrix} z^{\mu_1} & \cdots & z^{\mu_n} \\ \mu_1 z^{\mu_1 - 1} & \cdots & \mu_n z^{\mu_n - 1} \\ & \vdots & \\ (\cdots) z^{\mu_1 - n + 1} & & (\cdots) z^{\mu_n - n + 1} \end{vmatrix} = (?)\, z^{\sum_{j=1}^{n} (\mu_j - (j-1))}$$

in fact, non zero.

(uses $\mu_1 < \dots < \mu_n$)

<u>defn</u> the weight of z with respect to A is defined to be $\tau(z) = \sum_{j=1}^{n} (\mu_j - j + 1)$

prop The Wronskian of $\varphi_1, \cdots, \varphi_n$ has order $\tau(z)$ at z

It is easy to see that is is true for any basis of A.

pf: $W(\varphi_1, \cdots \varphi_n) = \det \begin{pmatrix} \varphi_1 & \cdots & \varphi_n \\ \varphi_1' & \cdots & \varphi_n' \\ & & \\ \varphi_1^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{pmatrix}$

i) f: holomorphic on D

$W(f\varphi_1, \cdots, f\varphi_n) = f^n W(\varphi_1, \cdots, \varphi_n)$ ⟵ chain rule & property of determinants

$$\begin{bmatrix} f\varphi_1 & f\varphi_2 & \cdots \\ f\varphi_1' + \cancel{f'\varphi_1} & f\varphi_2' + \cancel{f'\varphi_2} & \cdots \\ \cdots & & \end{bmatrix} \times \left(-\frac{f'}{f}\right)$$

ii) induction : $n = 1$ trivial

$$\operatorname{ord}_z W(\varphi_1, \cdots, \varphi_n) = \operatorname{ord}_z \Big(\varphi_1^{n+1} W(1, \underbrace{\varphi_2/\varphi_1, \cdots, \varphi_n/\varphi_1}_{\text{holomorphic near } z}) \Big)$$

$$= \mu_1(n+1) + \operatorname{ord}_z W((\varphi_2/\varphi_1)', \cdots, (\varphi_n/\varphi_1)')$$

induction hypothesis $\rightarrow$ $= \mu_1(n+1) + \sum_{j=2}^{n} \big((\mu_j - \mu_1 - 1) - (j-2) \big)$. #

<u>Cor</u> $W(\varphi_1, \cdots, \varphi_n) \equiv 0$ if $\{\varphi_1, \cdots, \varphi_n\}$: linearly dependent (on D)

V : n-dim $\Rightarrow$ $W(\varphi_1, \cdots, \varphi_n)$ is a holomorphic function which is NOT identically zero.

$\Rightarrow$ zeros are discrete

$\leftrightarrow$ z with $\mu(z) > 0$

conclusion on the Weierstrass points

NON Weierstrass points :

n	0	1	$\cdots$	$g-1$	g	$g+1$	$\cdots$	$2g-2$	$2g-1$	$2g$
$r(P^{-n})$	1	1	$\cdots$	1	1	2		$g-1$	g	$g+1$
$i(P^n)$	g	$g-1$	$\cdots$	1	0	0		0	0	0

Weierstrass point $\Leftrightarrow$ $i(P^g) > 0$ $\Leftrightarrow$ $r(P^{-g}) \geq 2$

$\Leftrightarrow$ the coefficient functions of the holomorphic differentials have positive weight at P

i) Note that when $g = 1$. $i(P) = r(K^{-1}P) = 0$ $\Rightarrow$ NO Weierstrass point

$\deg = 2g - 2 = 0$ $\Rightarrow$ $\deg = 1$

$\Rightarrow$ Assume $g \geq 2$

ii) global way of the Wronskian

$\mathcal{H}$ = holomorphic differentials, basis $\{\zeta_1, \cdots, \zeta_g\}$

in terms of local coordinate $\zeta_j = \varphi_j(z)\,dz$

$\rightsquigarrow$ look at the zeros of $W(\varphi_1, \cdots, \varphi_g)$ to find the Weierstrass points

in another coordinate $\zeta_j = \tilde{\varphi}_j(w)\,dw$ $\quad \tilde{\varphi}_j(w)\frac{dw}{dz} = \varphi_j(z)$ on the overlap region

$$W(\varphi_1, \cdots, \varphi_g) = \det\begin{pmatrix} \varphi_1(z) & \cdots \\ \frac{d}{dz}\varphi_1(z) & \\ \vdots & \\ \frac{d^{g-1}}{dz^{g-1}}\varphi_1(z) & \end{pmatrix} = \left(\frac{dw}{dz}\right)^g \det\begin{pmatrix} \tilde{\varphi}_1(w) & \\ \frac{d}{dz}\tilde{\varphi}_1(w) & \cdots \\ \vdots & \\ \frac{d^{g-1}}{dz^{g-1}}\tilde{\varphi}_1(w) & \end{pmatrix}$$

but $\frac{d}{dz}\tilde{\varphi}_1(w) = \left(\frac{d}{dw}\tilde{\varphi}_1(w)\right)\frac{dw}{dz}$ $\qquad \frac{(1+g-1)\,g-1}{2} = \frac{g(g-1)}{2}$ $\qquad g + \frac{g(g-1)}{2} = \frac{g(g+1)}{2}$

$$\Rightarrow \underbrace{W(\varphi_1, \cdots, \varphi_g)}_{\text{in } z} = \left(\frac{dw}{dz}\right)^{\frac{g(g+1)}{2}} \underbrace{W(\tilde{\varphi}_1, \cdots, \tilde{\varphi}_g)}_{\text{in } w}$$

"$\Rightarrow$" $\quad W(\varphi_1, \cdots, \varphi_g)(dz)^{+\frac{g(g+1)}{2}} = W(\tilde{\varphi}_1, \cdots, \tilde{\varphi}_g)(dw)^{+\frac{g(g+1)}{2}}$ on the overlap region

<u>defn</u> $q \in \mathbb{Z}$. a holomorphic q-differential consists of $f_\alpha(z_\alpha)$ on each U_α such that $f_\beta(z_\beta \circ z_\alpha^{-1})\left(\frac{dz_\beta}{dz_\alpha}\right)^q = f_\alpha(z_\alpha)$ if $U_\alpha \cap U_\beta \neq \phi$

and denote it by $f_\alpha(z_\alpha)(dz_\alpha)^q$

<u>rmk</u> the above Wronskian is a holomorphic $(+\frac{g(g+1)}{2})$-differential

iii) holomorphic q-differential $\Longleftrightarrow$ meromorphic function

- Fix a (non-trivial) holomorphic differential ω
  $\Rightarrow$ ω^q is a q-differential
- μ: holomorphic q-differential $\Rightarrow$ $\frac{\mu}{\omega^q}$: meromorphic function
  can still associate its divisor
  $(\mu) \geq \underline{1}$ holomorphic condition $\qquad f = \frac{\mu}{\omega^q} \quad (\mu) \geq 0 \Leftrightarrow (f) \geq (\omega)^q$

$\{$holomorphic q-differential$\} \cong L((\omega)^q)$

$$\deg(\mu) = \deg(f) + \deg(\omega)^q = +q(2g-2)$$

For $q = +\frac{g(g+1)}{2}$ $\qquad \deg = g(g+1)(g-1)$

<u>thm</u> when $g \geq 2$, $\sum_{P \in M} z(P) = g(g+1)(g-1) = g^3 - g$

iv) For a Weierstrass point

$$z(P) = \sum_{j=1}^{g}(\mu_j - j + 1) = \sum_{j=1}^{g}(n_j - j) \qquad n_j\text{: gaps} \Leftrightarrow i(P^{n_j}) = i(P^{n_j - 1}) + 1$$

$$= (1+2g)g - \sum_{j=1}^{g}\alpha_j - \frac{(1+g)g}{2} \leq \left(\frac{3}{2}g^2 + \frac{g}{2}\right) - (g(g-1) + 2g)$$
$$= \frac{1}{2}g(g-1)$$

<u>Cor</u> $2(g+1) \leq \#\{\text{Weierstrass point}\} \leq g^3 - g$