divisor & Riemann-Roch [FK; §III.4]

divisor : a convenient way to record the zeros & poles

[in general (algebraic geometry), divisor means (complex) codimension one objects ⇝ zeros of some function

mutiply / divide the function ⟺ change the zero loci]

Suppose that M is a compact Riemann surface

defn a divisor on M consists of some finite point with weight

denote it by $\mathfrak{A} = P_1^{\alpha_1} \cdots P_k^{\alpha_k}$ $\quad \alpha_j \in \mathbb{Z}$ $\quad (P_i P_j = P_j P_i)$

$= \prod_{P \in M} P^{\alpha(P)}$ $\quad \alpha(P) \in \mathbb{Z}$, nonzero for only finitely many $P \in M$

$\mathrm{Div}(M) = \{\text{all divisors}\}$ is an abelian group

$$\mathfrak{A} \cdot \mathfrak{B} = \Big(\prod_{P\in M} P^{\alpha(P)}\Big) \cdot \Big(\prod_{P\in M} P^{\beta(P)}\Big) = \prod_{P\in M} P^{\alpha(P)+\beta(P)}$$

The identity is the element with $\alpha(P) = 0 \quad \forall P$ and is denoted by 1

$$\mathfrak{A}^{-1} = \prod_{P\in M} P^{-\alpha(P)}$$

rmk modern notation in algebraic geometry : $\sum_{P\in M} \alpha(P) P$, using addition

- We can define the total degree of a divisor by

$$\deg : \mathrm{Div}(M) \to \mathbb{Z} \qquad \text{(group homomorphism)}$$
$$\prod P^{\alpha(P)} \longrightarrow \sum_{P\in M} \alpha(P)$$

- $\mathcal{M}(M) = \{\text{meromorphic function on } M\}$

$\mathcal{M}(M)^* = \mathcal{M}(M) \setminus \{\text{constant zero}\}$: group, binary operation is the usual multiplication

$$\mathcal{M}(M)^* \to \mathrm{div}(M)$$
$$f \longmapsto (f) = \prod_{P\in M} P^{\mathrm{ord}_P f}$$

what is the degree of this divisor ?

$\deg(f) = \deg f - \deg f$

degree of the divisor (f)

degree of $f : M \to \hat{\mathbb{C}}$

total order of 0 & ∞

- the image of the above map : principal divisor

clearly, $\deg : \mathrm{Div}(M) / \{\text{principal ones}\} \longrightarrow \mathbb{Z}$

the divisor class group

- similarly, for a meromorphic differential ω

we can consider $(\omega) = \prod_{P\in M} P^{\mathrm{ord}_P \omega}$

It is clear that $(\omega) = (\tilde{\omega}) \cdot \Big(\frac{\omega}{\tilde{\omega}}\Big)$ $\quad$ (ω) defines a class in $\mathrm{Div}(M) / \{\text{principal}\}$

$\in \mathcal{M}(M)^*$

This class in $\mathrm{Div}(M)/\{\text{principal}\}$ is called the canonical class

deg $= 2g-2$ $\boxed{HW}$ by using Riemann Roch

compare the order of zeros & poles : partial order

<u>def</u> $\mathcal{A} \geq \mathcal{B}$ if $\alpha(p) \geq \beta(p)$ $\forall p$

$$\mathcal{A} = \prod_{P\in M} P^{\alpha(P)}, \quad \mathcal{B} = \prod_{P\in M} P^{\beta(P)}$$

<u>e.g.</u> $\mathcal{A} = P\cdot Q^{-1}$ $\quad (f) \geq \mathcal{A}$

$$\Leftrightarrow \begin{cases} \mathrm{ord}_P f \geq 1 \\ \mathrm{ord}_Q f \geq -1 \\ \mathrm{ord}_R f \geq 0 & R\in M\setminus\{P,Q\} \end{cases} \Leftrightarrow \begin{cases} f: \text{holomorphic on } M\setminus Q \\ \text{vanishes at } P \\ \text{at worst a simple pole at } Q \end{cases}$$

certain subspace of meromorphic functions

<u>defn</u> $\mathcal{A}$: divisor

let $L(\mathcal{A}) = \{ f \in \mathcal{M}(M) \mid (f) \geq \mathcal{A} \}$

$\boxed{\text{check}}$ $L(\mathcal{A})$ is a finite dimensional vector space

no nontrivial holomorphic function on M

denote by $r(\mathcal{A})$ the dimension of $L(\mathcal{A})$

<u>prop</u> i) if $\mathcal{A} \geq \mathcal{B}$, $L(\mathcal{A}) \subseteq L(\mathcal{B})$

ii) $L(1) = \mathbb{C} = \{\text{constant functions}\}$

iii) if $\deg \mathcal{A} > 0$, the $L(\mathcal{A}) = \{0\}$

pf: $(f) \geq \mathcal{A} \geq \mathcal{B}$ #

certain subspace of meromorphic differentials

<u>defn</u> $\Omega(\mathcal{A}) = \{ \omega: \text{meromorphic differential} \mid (\omega) \geq \mathcal{A} \}$

$\boxed{\text{check}}$ $\Omega(\mathcal{A})$ is a finite dimensional vector space

Denote by $i(\mathcal{A})$ the dimension of $\Omega(\mathcal{A})$

$\dim \mathcal{H} = g$

<u>prop</u> $\Omega(1) = \mathcal{H} \Rightarrow i(1) = g$

<u>thm</u> i) $r(\mathcal{A})$ & $i(\mathcal{A})$ only depends on $[\mathcal{A}] \in \mathrm{Div}(M)/\text{principal}$

ii) (Serre duality). $r(\mathcal{A}(\omega)^{-1}) = i(\mathcal{A})$ for any nonzero meromorphic differential ω

pf: i) $[\mathcal{A}] = [\mathcal{B}] \Leftrightarrow \mathcal{A} = \mathcal{B}(f)$ for some $f \in (\mathcal{M}(M))^*$

$g \in L(\mathcal{A}) \Leftrightarrow (g) \geq \mathcal{A} \Leftrightarrow (\frac{g}{f}) \geq \mathcal{B} \Leftrightarrow \frac{g}{f} \in L(\mathcal{B})$

$\Rightarrow r(\mathcal{A}) = r(\mathcal{B})$. same argument for $i(\mathcal{A}) = i(\mathcal{B})$

ii) $f \in L(\mathcal{A}(\omega)^{-1}) \Leftrightarrow (f) \geq \mathcal{A}(\omega)^{-1}$

$\Leftrightarrow (f\omega) \geq \mathcal{A} \Leftrightarrow f\omega \in \Omega(\mathcal{A})$ #

Riemann-Roch theorem (for $\mathcal{A} \geq 1$)

thm $g = \text{genus}(M)$, $\mathcal{A} \geq 1$ → true for any divisor. takes a little more effort to do the general case

$$r(\mathcal{A}^{-1}) = \deg \mathcal{A} + 1 - g + i(\mathcal{A})$$

pf: $\mathcal{A} = P_1^{n_1} \cdots P_m^{n_m}$. $\mathcal{A} \geq 1 \iff n_j \geq 0 \quad \forall j$

$L(\mathcal{A}^{-1}) = \{ \text{meromorphic function } f \mid (f) \geq \mathcal{A}^{-1} \}$

holo on $M \setminus \{P_1, \dots, P_m\}$ at worst order n_k at P_k

$\mathcal{A}' = P_1^{n_1+1} \cdots P_m^{n_m+1}$

$\Omega_0((\mathcal{A}')^{-1}) = \{ \text{meromorphic differential } \omega \mid \int_{A_j} \omega = 0,\ \mathrm{Res}_{P_k} \omega = 0,\ \omega \geq (\mathcal{A}')^{-1} \}$

holo on $M \setminus \{P_1, \dots, P_m\}$ at worst order n_k+1 at P_k

zero z^{-1}-term

$$\begin{tikzcd} L(\mathcal{A}^{-1}) \arrow[r, "d"] & \Omega_0((\mathcal{A}')^{-1}) \arrow[r, "S"] & \mathbb{C}^{\deg \mathcal{A}} \arrow[d, shift right, "T^t"'] \\ & & \mathcal{H} \arrow[u, shift right, "T"'] \end{tikzcd}$$

$\ker d = \{\text{constants}\}$

S: coefficient in the singular part of the Laurent series at $\{P_k\}_{k=1}^m$

T: $\theta \mapsto$ coefficient of the Taylor expansion: $1, \dots, z^{n_k - 1}$ at P_k

same as last time → bilinear relation $\Rightarrow$ $\mathrm{im}(S \circ d) = \ker(T)^{\dagger}$

$$\begin{aligned} r(\mathcal{A}^{-1}) &= \dim L(\mathcal{A}^{-1}) = \dim(\mathrm{im}(S \circ d)) + 1 \\ &= \mathrm{nullity}(T^{\dagger}) + 1 = \deg(\mathcal{A}) - \mathrm{rank}(T^{\dagger}) + 1 \\ &= \deg(\mathcal{A}) - \mathrm{rank}(T) + 1 \\ &= \deg(\mathcal{A}) + g + 1 - \mathrm{nullity}(T) \end{aligned}$$

$\ker(T) = \Omega(\mathcal{A}) = \{ \omega \mid \omega \geq (\mathcal{A}) \}$ ✱

Cor if $g = 0$, $M \cong S^2$ (biholomorphically)

pf: $P \in M$. $\ell(P^{-1}) = 1 + 0 - 1 - i(P)$

but $\Omega(P) \subset \Omega(1) \Rightarrow i(P) \leq i(1) = 0$

$\Rightarrow \ell(P^{-1}) = 2 \qquad (\ell(1) = 1)$

$\Rightarrow \exists$ meromorphic function with only one pole ✱

rmk it is nothing more than a fancy way to do the HW.

Cor the Riemann-Roch theorem holds for $\mathcal{A}$ provided

i) $\mathcal{A} \sim \mathcal{B}$ for some $\mathcal{B} \geq 1$ ($\sim$: differ by a principal divisor)

ii) $\mathcal{Z}$: a canonical divisor, $\mathcal{Z}/\mathcal{A} \sim \mathcal{B}$ for some $\mathcal{B} \geq 1$

pf: i) $\ell, i, \deg$ are the same under $\sim$ $\quad (\mathcal{A}\mathcal{B} \sim \mathcal{Z})$

ii) $r(\mathcal{B}^{-1}) - i(\mathcal{B}) = \deg \mathcal{B} + 1 - g$

Serre duality: $i(\mathcal{A}) = r(\mathcal{A}\mathcal{Z}^{-1}) = r(\mathcal{B}^{-1})$

$i(\mathcal{B}) = r(\mathcal{A}^{-1})$

$\Rightarrow i(\mathcal{A}) - r(\mathcal{A}^{-1}) = \deg \mathcal{Z}/\mathcal{A} + 1 - g = \deg \mathcal{Z} - \deg \mathcal{A} + 1 - g$

$\deg \mathcal{Z} = 2g - 2$

✱

Riemann-Roch theorem (remaining case)

$$r(\mathcal{A}^{-1}) = \deg \mathcal{A} + 1 - g + i(\mathcal{A})$$

We have shown that it is true for $\mathcal{A} \geq 1$. $\mathcal{A}$ or $Z\mathcal{A}^{-1}$ (canonical) is equivalent to $\mathcal{B}$ with $\mathcal{B} \geq 1$

- $\mathcal{A} \sim \mathcal{B} \geq 1 \iff \mathcal{A}(f) \geq 1$ for some $f \in \mathcal{M}^*(M)$
  $\iff (f) \geq \mathcal{A}^{-1} \iff r(\mathcal{A}^{-1}) \geq 1$
- $Z\mathcal{A}^{-1} \sim \mathcal{B} \geq 1 \iff (\omega)\mathcal{A}^{-1} \geq 1$ for some nonzero meromorphic differential ω
  $\iff (\omega) \geq \mathcal{A} \iff i(\mathcal{A}) \geq 1$

$\Rightarrow$ if neither $\mathcal{A}$ or $Z\mathcal{A}^{-1}$ is equivalent to some $\mathcal{B}$ with $\mathcal{B} \geq 1$
then $r(\mathcal{A}^{-1}) = 0 = i(\mathcal{A})$

Hence, it remains to prove the following claim

<u>claim</u>: if neither $\mathcal{A}$ or $Z\mathcal{A}^{-1}$ is equivalent to some $\mathcal{B}$ with $\mathcal{B} \geq 1$
then, $\deg \mathcal{A} = g-1$ $(= \deg Z\mathcal{A}^{-1})$

pf: 1° $\deg \mathcal{A} < g$? Suppose NOT.

Write $\mathcal{A} = \mathcal{A}_1 / \mathcal{A}_2$ $\quad \mathcal{A}_1$ & $\mathcal{A}_2$: coprime (no common points)
$\mathcal{A}_1, \mathcal{A}_2 \geq 1$

$\Rightarrow \mathcal{A}_1 = \mathcal{A}\mathcal{A}_2 \geq \mathcal{A} \quad \Rightarrow \quad L(\mathcal{A}^{-1}) \subset L(\mathcal{A}_1^{-1})$

$\Rightarrow r(\mathcal{A}^{-1}) \geq r(\mathcal{A}_1^{-1}) - \deg \mathcal{A}_2$

can apply R-R

impose vanishing @ $\mathcal{A}_2$-conditions
$\deg \mathcal{A}_2$ linear equation

By R-R (or Riemann inequality)

$r(\mathcal{A}_1^{-1}) \geq \deg \mathcal{A}_1 + 1 - g = \deg \mathcal{A} + \deg \mathcal{A}_2 + 1 - g \geq 1 + \deg \mathcal{A}_2$

hypothesis

Thus, $r(\mathcal{A}^{-1}) \geq 1 \Rightarrow \mathcal{A} \sim \mathcal{B} \geq 1 \rightarrow\leftarrow$

2° similarly, $\deg Z\mathcal{A}^{-1} < g \Rightarrow \deg \mathcal{A} > g-2$ $\divideontimes$