M: compact Riemann surface. $H_1(M) \cong \mathbb{Z}^{2g}$ with intersection pairing

closed loops which do not bound $\longleftrightarrow$ obstruction for closed 1-form to being exact

Space of all harmonic differentials [FK, § III.2]

1° $L^2(M) = H \oplus \underset{\{df\}}{\underset{\shortparallel}{E}} \oplus \underset{\{*df\}}{\underset{\shortparallel}{E^*}}$

Now, suppose there is a ω: C^1. $d\omega = 0$ and γ: cycle on M (basically, directed loops)

such that $\int_\gamma \omega \neq 0$

$\Rightarrow \omega = \omega_h + df + 0$ (since $d\eta = 0$)
$\quad\; H \oplus E \oplus E^*$

if $\omega_h = 0$

$\Rightarrow 0 \neq \int_\gamma \omega = \int_\gamma df = 0$ $\rightarrow\leftarrow$

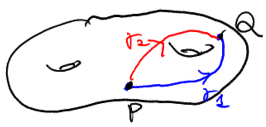
thus, $\omega_h \neq 0$ and $\int_\gamma \omega_h = \int_\gamma \omega \neq 0$

2° Let $g = \text{genus}(M)$, fix a canonical homology basis $\{A_1, \cdots, A_g, B_1, \cdots, B_g\}$

Consider $H \xrightarrow{\Pi} \mathbb{C}^{2g}$ (the period map)
$\omega \longmapsto (\int_{A_1} \omega, \cdots$

thm the above map is an isomorphism, and thus $H \cong \mathbb{C}^{2g}$

pf: (injectivity) if $\Pi(\omega) = 0$, fix $P \in M$.



$\forall Q \in M$, define $f(Q) = \int_\gamma \omega$ γ: any directed curve connecting P & Q

Since $\Pi(\omega) = 0 \Rightarrow f$ is well-defined

By construction, $df = \omega$. $\Rightarrow \omega \in H \cap E = 0$

(surjectivity) Consider the Poincaré dual of A_1: η_{A_1}.

$\Rightarrow \int_{B_1} \eta_{A_1} = -1$ and $\int_{A_k} \eta_{A_1} = 0 = \int_{B_{k\neq 1}} \eta_{A_1}$

By 1°, $\eta_{A_1} = \underset{H}{\underset{\Uparrow}{\omega_1}} + (\cdots)$ and $\Pi(\omega_1) = (0, \cdots, 0, -1, \cdots, 0) \in \mathbb{C}^{2g}$

Similar discussion proves the surjectivity $\#$

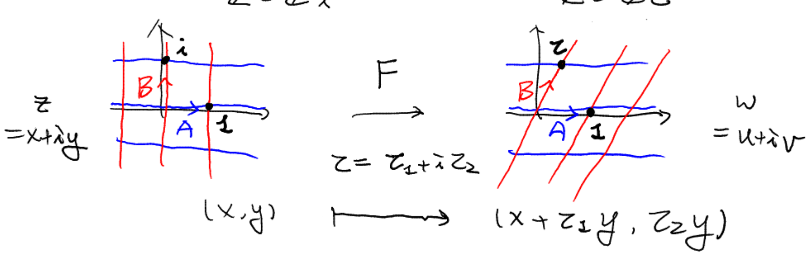
In other words, nontrivial loops + nontrivial intersection relations
$\Rightarrow$ existence of harmonic differential.

What about holomorphic differentials?

digression: T^2 (compare with the last exercise of HW 1)

fix $\tau \in \mathbb{H}$, not in $\mathbb{Z} \oplus \mathbb{Z}i$

$\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}i$ $\qquad$ $\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$ $\qquad$ different or same?



Same in the sense that F descends to define a diffeomorphism

harmonic differential

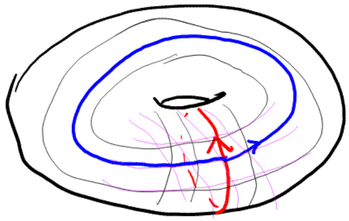
$dx, dy \longleftrightarrow du - \frac{\tau_1}{\tau_2} dv, \; \frac{1}{\tau_2} dv$

different in the sense of Riemann surface structure

holomorphic differential

dz | dw

$\int_A dz = 1,\ \int_B dz = i$ | $\int_A dw = 1,\ \int_B dw = \tau$



$\Rightarrow$ holomorphic differential (using $*$) can detect the Riemann surface structure

holomorphic differential and period integral

genus $(M) = g \quad \Rightarrow \quad H \cong \mathbb{C}^{2g}$

Denote by $\mathcal{H} = \{$ holomorphic differentials $\}$

$f(z)\,dz$ ← holomorphic

conjugation

$\overline{f(z)}\,d\bar{z}$: still harmonic (anti-holomorphic)

clearly, $\mathcal{H} \oplus \overline{\mathcal{H}} \subset H$ and $\dim \mathcal{H} = \dim \overline{\mathcal{H}}$

on the other hand, given $\alpha \in H \Rightarrow \alpha + i*\alpha \in \mathcal{H}$ & $\alpha - i*\alpha \in \overline{\mathcal{H}}$

$f\,dz + g\,d\bar{z} \quad \Rightarrow \quad 2f\,dz \qquad 2g\,d\bar{z}$

$d\alpha = 0 = d*\alpha \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 = \frac{\partial g}{\partial z}$

$\alpha = \frac{1}{2}(\alpha + i*\alpha) + \frac{1}{2}(\alpha - i*\alpha) \Rightarrow \mathcal{H} \oplus \overline{\mathcal{H}} \supset H$

Prop $\mathcal{H} \cong \mathbb{C}^g$

Q What can we say about their image under the period map?

$\{A_1, \cdots, A_g, B_1, \cdots, B_g\}$ canonical basis for $H_1(M)$

$\rightsquigarrow \{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\}$ basis for H

$$\begin{cases} \alpha_j = \eta_{B_j} + d(C^\infty) \\ \beta_j = -\eta_{A_j} + d(C^\infty) \end{cases}$$

$\Rightarrow$ need to study the period map of $*\alpha_j$ and $*\beta_j$

Given any ω, $d\omega = 0 \Rightarrow \int_{A_j} \omega = -\iint_M \omega \wedge \eta_{A_j} = \iint_M \omega \wedge \beta_j + \cancel{\iint_M \omega \wedge d(\cdots)}^{\,0}$

$\int_{B_j} \omega = -\iint_M \omega \wedge \alpha_j$

Hence, we need to study $\iint \alpha_j \wedge *\alpha_k$, $\iint \alpha_j \wedge *\beta_k$, $\iint \beta_j \wedge *\beta_k$

- By construction, η_{B_j} is real, i.e. $\eta_{B_j} = \overline{\eta_{B_j}} \Rightarrow \alpha_j = \overline{\alpha_j}$
  Hence, the integrals have real value.
- $\iint \alpha_j \wedge *\alpha_k = \iint *\alpha_j \wedge *(*\alpha_k) = \iint *\alpha_j \wedge (-\alpha_k) = \iint \alpha_k \wedge *\alpha_j$
  $\iint \alpha_j \wedge *\beta_k = \iint \beta_k \wedge *\alpha_j \quad . \quad \iint \beta_j \wedge *\beta_k = \iint \beta_k \wedge *\beta_j$

$$\Gamma = \begin{array}{c c} & \begin{array}{cc} *\alpha_k & *\beta_k \end{array} \\ \begin{array}{c} \alpha_j \\ \beta_j \end{array} & \begin{bmatrix} \Gamma^1 & \Gamma^2 \\ \Gamma^3 = (\Gamma^2)^t & \Gamma^4 \end{bmatrix} \end{array} \qquad \begin{aligned} \Gamma^1_{jk} &= \iint \alpha_j \wedge *\alpha_k & \Gamma^2_{jk} &= \iint \alpha_j \wedge *\beta_k \\ \Gamma^3_{jk} &= \iint \beta_j \wedge *\alpha_k & \Gamma^4_{jk} &= \iint \beta_j \wedge *\beta_k \end{aligned}$$

Prop Γ : real, symmetric matrices. and $\Gamma > 0$ (positive definite)

pf: Given any $c_j, d_j \in \mathbb{C}$

$0 < \| c_j\alpha_j + d_j\beta_j \|^2 = c_j \bar{c}_k \iint \alpha_j \wedge *\bar{\alpha}_k + \cdots = [c_1 \cdots c_g\ d_1 \cdots d_g]\, \Gamma \begin{bmatrix} \bar{c}_1 \\ \vdots \\ \bar{d}_g \end{bmatrix}$

$\#$

- Express $*\alpha_k$ and $*\beta_k$ in terms of $\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\}$

$$*\alpha_k = \sum_{j=1}^{g} \left(\lambda_{kj}\alpha_j + \lambda_{k(g+j)}\beta_j\right)$$
$$*\beta_k = \sum_{j=1}^{g} \left(\lambda_{(k+g)j}\alpha_j + \lambda_{(k+g)(g+j)}\beta_j\right)$$

$$\left.\begin{aligned}
\lambda_{kj} &= \int_{A_j} *\alpha_k = \iint *\alpha_k \wedge \beta_j = -\Gamma^3_{jk} = -\Gamma^2_{kj} \\
\lambda_{k(g+j)} &= \int_{B_j} *\alpha_k = -\iint *\alpha_k \wedge \alpha_j = \Gamma^1_{jk} = \Gamma^1_{kj} \\
\lambda_{(k+g)j} &= \int_{A_j} *\beta_k = \iint *\beta_k \wedge \beta_j = -\Gamma^4_{jk} = -\Gamma^4_{kj} \\
\lambda_{(k+g)(j+g)} &= \int_{B_j} *\beta_k = -\iint *\beta_k \wedge \alpha_j = \Gamma^2_{jk} = (\Gamma^2)^*_{kj}
\end{aligned}\right\} \Rightarrow \underline{\lambda = \begin{bmatrix} -\Gamma^2 & \Gamma_1 \\ -\Gamma_4 & (\Gamma^2)^* \end{bmatrix}}$$

(matrix representation of $*$)

- Since $*^2 = -\mathrm{Id}$ $\quad \begin{bmatrix} -\Gamma^2 & \Gamma^1 \\ -\Gamma^4 & (\Gamma^2)^* \end{bmatrix} \begin{bmatrix} -\Gamma^2 & \Gamma^1 \\ -\Gamma^4 & (\Gamma^2)^* \end{bmatrix} = -I_{2g}$

$$\Rightarrow \underline{(\Gamma^2)^2 + \Gamma^1\Gamma^4 = -I_g}, \quad \underline{\Gamma^4\Gamma^2 = (\Gamma^2)^*\Gamma^4}, \quad \underline{\Gamma^2\Gamma^1 = \Gamma^1(\Gamma^2)^*}$$

prop $\{\alpha_1 + i*\alpha_1, \cdots, \alpha_g + i*\alpha_g\}$ constitutes a basis for the holomorphic differentials

pf: Consider the period map $\xrightarrow{\Pi} \mathbb{C}^{2g}$

$$\int_{A_k} \alpha_j + i*\alpha_j = 1 - i\Gamma^2_{jk} \qquad \int_{B_k} \alpha_j + i*\alpha_j = 0 + i\Gamma^1_{jk}$$

$$\Rightarrow \quad \begin{array}{c|c|c} & \int_{A_k} - & \int_{B_k} - \\ \alpha_j + i*\alpha_j & I - i\Gamma^2 & i\Gamma^1 \end{array} \longrightarrow \text{rank } g$$

※

Change of basis, invert either $I - i\Gamma^2$ or $i\Gamma^1$ ↖ easeier $\quad -i(\Gamma^1)^{-1}$

Take $-i\sum_{\ell} (\Gamma^1)^{-1}_{j\ell}\,(\alpha_\ell + i*\alpha_\ell)$

$$\xrightarrow{\int_{A_k}} -i(\Gamma^1)^{-1}_{j\ell}(I + i\Gamma^2)_{\ell k} = -i(\Gamma^1)^{-1} + (\Gamma^1)^{-1}\Gamma^2$$
$$\xrightarrow{\int_{B_k}} -i(\Gamma^1)^{-1}_{j\ell}(i\Gamma^1)_{\ell k} = \delta_{jk}$$

$$\Big\downarrow \Pi$$

$$\left[-i(\Gamma^1)^{-1} + (\Gamma^1)^{-1}\Gamma^2 \;\middle|\; I\right]$$

It is more common to start with $\beta_j \rightsquigarrow \beta_j + *\beta_j$

Same computation

$$\begin{array}{c|c|c} & \int_{A_k} - & \int_{B_k} \\ \beta_j + i*\beta_j & -i\Gamma_4 & I + i(\Gamma^2)^* \end{array}$$

Change of basis by $i\sum_{\ell=1}^{g} (\Gamma_4)^{-1}_{j\ell}(\beta_\ell + i*\beta_\ell) \xrightarrow{\Pi} \left[I \;\middle|\; \underbrace{-(\Gamma^4)^{-1}(\Gamma^2)^* + i\Gamma_4^{-1}}_{\tau} \right]$

thm $\exists$ a unique basis for $\mathcal{H}$ $\{\zeta_1, \cdots, \zeta_g\}$ such that $\int_{A_k} \zeta_j = \delta_{kj}$

Moreover, $\left[\int_{B_k} \zeta_j\right]_{jk}$ has symmetric positive definite imaginary part

ω : closed 1-form $\quad \Pi(\omega) = [\int_{A_j} \omega \;\vdots\; \int_{B_j} \omega] \in \mathbb{C}^g \oplus \mathbb{C}^g = \mathbb{C}^{2g}$

A-periods $\qquad$ B-periods

prop Let θ be a holomorphic differential
i) If all the A-periods vanish, $\theta = 0$
ii) If all the periods are real, $\theta = 0$

pf: $\theta = \sum_j c_j \zeta_j \rightsquigarrow$ simple linear algebra argument #

period integral of meromorphic differentials [FK; § III. 3]

<u>classical terminology</u> $\quad$ abelian differential = meromorphic differential

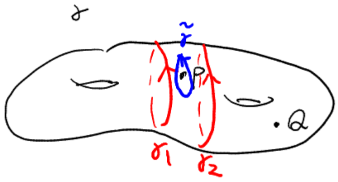
i) abelian differential of 1^{st} kind : holomorphic ones
ii) abelian differential of 2^{nd} kind : zero residues
iii) abelian differential of 3^{rd} kind : general ones

Since $\sum_{P \in M} \mathrm{Res}_P \tau = 0$, it suffices to consider

$$\begin{cases} \tau: \text{holomorphic on } M \setminus \{P, Q\} \\ \mathrm{ord}_P \tau = -1 = \mathrm{ord}_Q \tau \\ \mathrm{Res}_P \tau = 1, \; \mathrm{Res}_Q \tau = -1 \end{cases} \quad \text{and} \quad \begin{cases} \tau: \text{holomorphic on } M \setminus P \\ \mathrm{ord}_P \tau = -n < -1 \end{cases}$$

First, consider the case of two simple poles.

1° $\int_\gamma \tau$: makes sense ?



At least, γ cannot pass through τ

Secondly, $\int_{\gamma_2} \tau - \int_{\gamma_1} \tau = \int_{\tilde{\gamma}} \tau \quad$ (Stokes)

$= 2\pi i \, \mathrm{Res}_P \tau = 2\pi i$

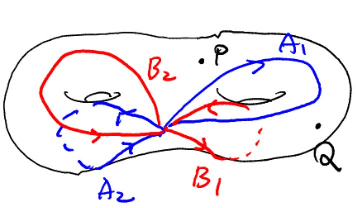
Hence, even if $[\gamma_2] = [\gamma_1]$ in $H_1(M)$

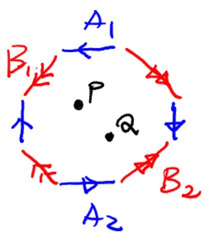
$\int_{\gamma_2} \tau$ might be different from $\int_{\gamma_1} \tau$

The singularity becomes the obstruction for the Stokes theorem argument

<u>rmk</u> correct cycle class : $H_1(M \setminus \{P, Q\})$

2° To proceed, fix a 2g-loops-cut such that P, Q do not belong to the loops cuts





$\{\zeta_j\}_{j=1}^g$: basis for $\mathcal{H}$

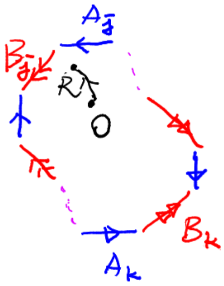
such that $\int_{A_k} \zeta_j = \delta_{jk}$

By adding a linear combination of $\{\zeta_j\}$ we may assume $\int_{A_j} \tau = 0 \quad \forall j$

(In other words, we are studying $\int_{A_j} \cdot \int_{B_j}$ for particular loops not for homology classes in $H_1(M)$)

goal evaluate $\int_{B_j} \tau$, somehow it shall relate to ζ_j

i) We can construct a partial anti-derivative for ζ_j



Fix $O \in$ interior (2g-polygon)

$\forall\ R \in$ 2g-polygon , define $f(R) = \int_O^R \zeta_j$

(f : holomorphic on $M \setminus \{\text{loops-cuts}\}$
NOT single valued on M)

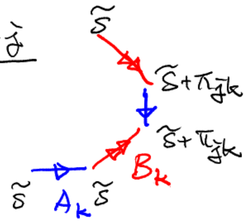
path inside the polygon

ii) the value of f on the vertices

$\int_{A_j} \zeta_j = 1$

$\int_{B_j} \zeta_j = \pi_{jj}$

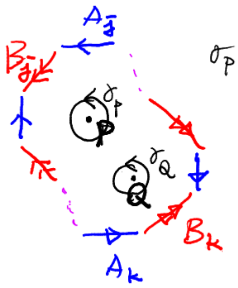
$\underline{k \neq j}$



$\int_{A_k} \zeta_j = 0$

$\int_{B_k} \zeta_j = \pi_{jk}$

iii)



$$\int_{\gamma_P \cup \gamma_Q} f \tau = 2\pi i \left(f(P)\, \mathrm{Res}_P \tau + f(Q)\, \mathrm{Res}_Q \tau \right)$$

argument principle

$df = \zeta_j$

$$= 2\pi i \left(f(P) - f(Q) \right) = 2\pi i \int_Q^P \zeta_j$$

$$= \int_{\partial(\text{4g-polygon})} f\tau$$

Stokes

for $k = j$

$A_j, -A_j \rightsquigarrow \pm \pi_{jj} \int_{A_j} \tau = 0$

$B_j, -B_j \rightsquigarrow$ left-over $= \int_{B_j} \tau$

for $k \neq j$

$A_k, -A_k \rightsquigarrow \pm \pi_{jk} \int_{A_k} \tau = 0$

$B_k, -B_k \rightsquigarrow$ cancel

iv) upshot $\int_{B_j} \tau = 2\pi i \int_Q^P \zeta_j$

For τ : holomorphic on $M \setminus \{P\}$, $\tau = \left(\frac{1}{z^n} + (\text{hol}) \right) dz$ on a nbd of P

The argument is completely parallel : still assume $\int_{A_j} \tau = 0$

$\zeta_j = df$ on the polygon , or $M \setminus \{\text{loops-cuts}\}$

$$\int_{\gamma_P} f\tau = \int_{B_j} \tau$$

same as above

$$\zeta_j \big|_{\text{nbd of } P} = \left(\sum_{k \geq 0} a_k^{(j)} z^k \right) dz \Rightarrow f \big|_{\text{nbd of } P} = \sum_{k \geq 0} \frac{a_k^{(j)}}{k+1} z^{k+1} + \text{Constant}$$

$$\Rightarrow \int_{\gamma_P} f\tau = 2\pi i\, \mathrm{Res}_P (f\tau) = 2\pi i\, \frac{a_{n-2}^{(j)}}{n-1}$$

$k = n-2$

potential

Given μ : meromorphic differential, when does there exist a meromorphic function $g \ni dg = \mu$

- if so, $g = \frac{a_{-n}}{z^n} + \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots$ near a pole

  $dg = -n a_{-n} \frac{1}{z^{n+1}} - \cdots - a_{-1} \frac{1}{z^2} + a_1 + 2a_2 z + \cdots$

  ↳ NO Residue

  $\Rightarrow \ \mathrm{Res}_p \mu = 0 \quad \forall p \in M$

- suppose that γ : directed closed curve on $M \setminus \{poles\}$

  $\Rightarrow \int_\gamma \mu = \int_\gamma dg = 0$

<u>prop</u> Fixed a $2g$ loops-cut on M : $\{A_1, \cdots, A_g, B_1, \cdots, B_g\}$

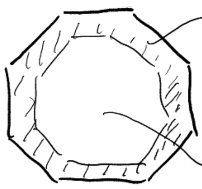
η : meromorphic differential, poles of η do not lie in the loops cut

Then, $\eta = dg$ if and only if $\begin{cases} \mathrm{Res}_p \eta = 0 & \forall p \\ \int_{A_j} \eta = 0 = \int_{B_j} \eta \end{cases}$

Pf: $\Rightarrow$) above

$\Leftarrow$) Define $g(Q) = \int_0^Q \eta$ any path not passing through the poles

well-definedness :



well-defined on this region (even after identification) since $\int_{A_j} \eta = 0 = \int_{B_j} \eta$

then, well-defined to the "interior" by using $\mathrm{Res}_p \eta = 0$ #

Riemann-Roch (simplest case)

i) Fix a loops-cut, $P \notin$ loops-cut, and $n \in \mathbb{N}$

Let $L(P^{-n}) = \{$ meromorphic functions on $M \mid$ holomorphic on $M \setminus P$, $\mathrm{ord}_p \geq -n \}$

↳ at worst $\frac{a_{-n}}{z^n} + \cdots + \frac{a_{-1}}{z} + \cdots$

<u>Goal</u> study how many of them, i.e. $\dim L(P^{n})$.

ii) $L(P^{-n}) \xrightarrow{d}$ certain meromorphic differentials

<u>idea</u> keep the conditions $\int_{A_j} \eta = 0$ and $\mathrm{Res}_p \eta = 0$

replace $\int_{B_j} \eta$ by the bilinear relation ← translate into some linear condition so that we can count the dimension

Let $\Omega_0(P^{-n-1}) = \{$ meromorphic differential on $M \mid$ holomorphic on $M \setminus P$, $\mathrm{Res}_p \eta = 0$, $\mathrm{ord}_p \eta \geq -(n+1)$, $\int_{A_j} \eta = 0 \}$

$\Rightarrow L(P^{-n}) \xrightarrow{d} \Omega_0(P^{-n-1})$

iii) $\ker d = \{$ constant functions $\}$

$\mathrm{im}\, d = \{ \eta \in \Omega_0(P^{-n-1}) \mid \int_{B_j} \eta = 0 \}$

$\dim L(P^{-n}) = \dim(\mathrm{im}\, d) + 1$ ← study $\dim(\mathrm{im}\, d)$ in step

$\geq \dim \Omega_0(P^{-n-1}) - g + 1$

iv) compare with $\Omega_0(P^{-n-1})$ and $\mathcal{H}$ $\xrightarrow{S \text{ (singular part)}}$ $\mathbb{C}^n$

$\eta = \sum_{k \geq -(n+1)} b_k z^k$ near P $\quad (b_{-(n+1)}, b_{-n}, \cdots, b_{-2})$

$b_{-1} = 0$

$\ker S = \Omega_0(P^{-n-1}) \cap \mathcal{H}$ but $\int_{A_j} \eta = 0$ and $\eta \in \mathcal{H} \Rightarrow \eta = 0$

Thus, S is injective

Also, the construction of meromorphic differential implies that S is surjective

$\Rightarrow \dim(\Omega_0(P^{-n-1})) = n \quad \Rightarrow \dim L(P^n) \geq n - g + 1$ —— Riemann inequality

v) Study im d carefully

$\eta = \sum_{k \geq -(n+1)} b_k z^k$ near P $\quad \Rightarrow \frac{1}{2\pi i} \int_{B_j} \eta = b_{-(n+1)} \frac{a_{n-1}}{n} + \cdots + b_{-2} \frac{a_0}{1}$

$b_{-1} = 0$

$\zeta_j = \sum_{k \geq 0} (a_k z^k)\, dz$ near P

vi) $L(P^{-n}) \xrightarrow{d} \Omega_0(P^{-n-1}) \xrightarrow[\simeq]{S} \mathbb{C}^n$ $\quad \sum_{j=1}^{g} c_j \left(\frac{a_{n-1}^{(j)}}{n}, \cdots, \frac{a_0^{(j)}}{1} \right)$

$\uparrow T \qquad\qquad \uparrow$

$\mathbb{C}^g \cong \mathcal{H}$ $\quad \sum_{j=1}^{g} c_j \zeta_j$

in matrix

$$T = \begin{array}{c} \\ n \end{array} \overset{g}{\begin{bmatrix} \frac{a_{n-1}^{(1)}}{n} & \cdots & \frac{a_{n-1}^{(g)}}{n} \\ \vdots & \ddots & \vdots \\ \frac{a_0^{(1)}}{1} & \cdots & \frac{a_0^{(g)}}{1} \end{bmatrix}}$$

$\frac{1}{2\pi i} \int_{B_j} \eta = 0 \quad \Rightarrow \operatorname{im}(S \circ d) = \ker(T^*) : \mathbb{C}^n \to \mathbb{C}^g$

$\Rightarrow \dim(\operatorname{im} d) = n - \operatorname{rank}(T^*)$

$= n - \operatorname{rank}(T)$

$= n - (g - \dim \ker(T))$

$\Rightarrow \dim L(P^{-n}) = n - g + 1 + \dim \ker(T)$ —— Riemann-Roch

$\ker(T)$ = holomorphic differential, vanishing at P to order at least n

rmk in fact $\ker(T) = 0$ for n: sufficiently large.