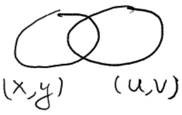
topology / homology of a compact Riemann surface [FK, §I.2]

(do the homotopy part later)

M: compact Riemann surface

- M is orientable. i.e. $\exists$ coordinate cover such that all the coordinate transitions have positive Jacobians

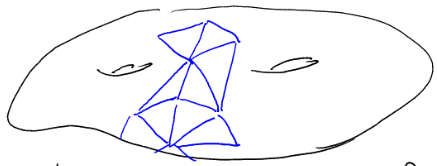


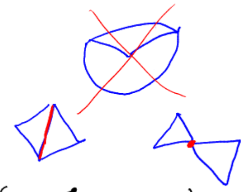
$$\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = \begin{vmatrix} u_x & -u_y \\ u_y & u_x \end{vmatrix} = u_x^2 + u_y^2 > 0$$

✪ classification: M must be a genus g surface

(More precisely, M is diffeomorphic to a genus g surface)

- M admits a triangulation:
(homeomorphic to a simplicial complex)

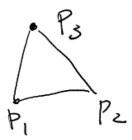




intersection of two Δ $\Rightarrow$ either ϕ, or a common face (/ or $\cdot$)
intersection of two / $\Rightarrow$ either ϕ, or a common vertex

(We may further assume each edge $\ni$ a smooth curve. and each $\Delta \subset$ some coordinate chart)

homology groups



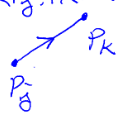
P_j: all vertices of a triangulation

abelian groups

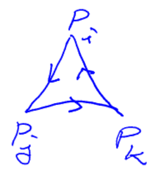
C_0: 0-chain $= \oplus \mathbb{Z} \langle P_j \rangle$

C_1: 1-chain $= \underset{\substack{\exists \text{edge} \\ \text{connecting } P_j, P_k}}{\oplus} \mathbb{Z} \langle P_j, P_k \rangle \Big/ \langle P_j, P_k \rangle = -\langle P_k, P_j \rangle$

$\langle P_j, P_k \rangle$: "directed" path



C_2: 2-chain $= \underset{\Delta}{\oplus} \mathbb{Z} \langle P_i, P_j, P_k \rangle \Big/ \langle P_i, P_j, P_k \rangle = \langle -P_j, P_i, P_k \rangle$



∂: boundary map
$$\begin{cases} \partial \langle P_j, P_k \rangle = P_k - P_j \\ \partial \langle P_i, P_j, P_k \rangle = \langle P_j, P_k \rangle - \langle P_i, P_k \rangle + \langle P_i, P_j \rangle \end{cases}$$

extend ∂ to C_* by linearity

$\downarrow \partial$

<u>lemma</u> $\partial^2 = 0$

$P_k - P_j - P_k + P_i - P_j - P_i = 0$

<u>goal</u> counting boundaryless objects which are not the boundary

$Z_n \subset C_n$ is the kernel of ∂
$\cup$
$B_n = \partial C_{n+1}$

$\Rightarrow H_n = Z_n / B_n$ the n-th homology group.

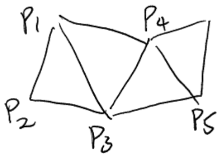
$$0 \to C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \to 0$$

For H_0: $C_0 = Z_0$. $B_0 = \partial C_1 = \oplus \mathbb{Z} (\langle P_k \rangle - \langle P_j \rangle)$

By connectedness, Σ coefficients $=$ even $\Leftrightarrow B_0$

$\Rightarrow C_0 / B_0 = \mathbb{Z}$ (generated by any point)

For H_2: $B_2 = 0$ $\quad Z_2 = \ker \partial = ?$



if the coefficient in $\langle P_1, P_2, P_3 \rangle \neq 0$

$\partial \mapsto \langle P_1, P_2 \rangle - \langle P_1, P_3 \rangle + \langle P_2, P_3 \rangle$

only way to cancel it $\Rightarrow$ include $\langle P_1, P_3, P_4 \rangle$ with the same coefficient

$\Rightarrow \ker Z_2$ is generated by the "surface itself" ↳ the fundamental class

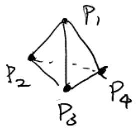
$\Rightarrow H_2 \cong \mathbb{Z}$

For H_1: discuss later $\cong \mathbb{Z}^{2g}$

rmk • H_* is independent of the construction of triangulation

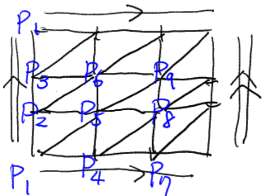
• in general, H_* could contain torsion part

e.g. • S^2 $\Rightarrow$ $H_1 = 0$



(any closed curve = ∂(some region))

• T^2



$\langle P_1, P_2 \rangle + \langle P_2, P_3 \rangle + \langle P_3, P_1 \rangle \xmapsto{\partial} 0$

and $\langle P_1, P_4 \rangle + \langle P_4, P_7 \rangle + \langle P_7, P_1 \rangle \xmapsto{\partial} 0$

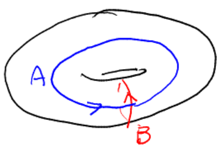
but they are not ∂(something) $\quad \Rightarrow H_1 = \mathbb{Z}^2$

roughly, $C_1 = \mathbb{Z}^{27} \xrightarrow{\partial} \mathbb{Z}^8 \subset \mathbb{Z}^9$

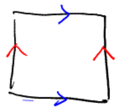
$\ker \cong \mathbb{Z}^{19}$

18-Δ's 1: redundant $\Rightarrow$ 17- relations $\qquad 19-17=2$

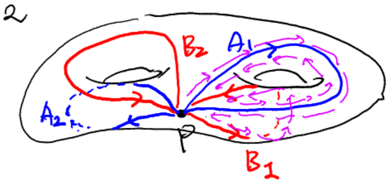
normal form



cut along A & B $\rightsquigarrow$



$g=2$



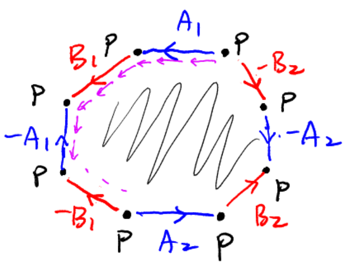
2g loops-cuts

i) With the same base point

ii) No non-contractible loop can be drawn, disjoint from any of the loops-cut

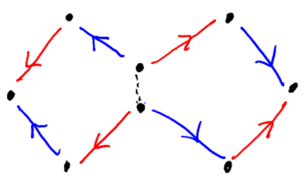
see $\rightarrow$ iii) In $M \setminus \{\text{loops-cuts}\}$, we can travel along its boundary so that boundary is always on the right

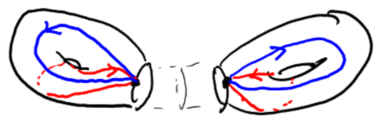
$M \setminus \{\text{loops-cuts}\}$



It is called the fundamental polygon

By identifying the boundaries, M can be recovered (as a topological space)





By constructing a triangulation based on these loop-cuts

$\Rightarrow H_1 = \mathbb{Z}^{2g}$ with generator corresponding to $\{A_1, \cdots A_g, B_1, \cdots, B_g\}$

Euler characteristic

$\chi(M) = V - E + F$ $\quad$ $V: \#\{\text{vertices}\}$ $\quad$ $E: \#\{\text{edges}\}$ $\quad$ $F: \#\{\Delta\text{'s}\}$

let $b_j = \operatorname{rank} H_j$ (as a $\mathbb{Z}$-module) $\quad$ Then, $\chi(M) = b_0 - b_1 + b_2$

pf: pretend they are vector spaces

$\xrightarrow{\partial} C_j \xrightarrow{\partial} C_{j-1}$ $\quad$ $\ker \partial = Z_j$. $\operatorname{im} \partial = B_j$. $\quad$ $\underset{\text{rank}}{\dim B_{j-1}} + \underset{\text{nullity}}{\dim Z_j} = \dim C_j$

$\dim H_j = \dim Z_j - \dim B_j$

$\Rightarrow \dim H_0 - \dim H_1 + \dim H_2 = \dim Z_0 - \underline{\dim B_0 - \dim Z_1} + \underline{\dim B_1 - \dim Z_2} - \cancel{\dim B_2}^{\,0}$

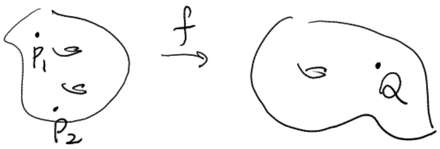
$= \dim C_0 - \dim C_1 + \dim C_2$

#

Riemann-Hurwitz formula

$M_1 \xrightarrow{f} M_2$ $\quad$ M_1, M_2: compact Riemann surfaces $\quad$ f: holomorphic map

$\Rightarrow$ recall $\deg f = \sum_{P \in f^{-1}(Q)} \underbrace{(b_f(P)+1)}_{\text{multiplicity at } P} \in \mathbb{N}$ (independent of Q)



$b_f(P) > 0$ if $f'(P) = 0$ (in local coordinate)

$\Rightarrow$ only finite points could have $b_f > 0$.

Q relating $\deg f$, $\chi(M_1)$, $\chi(M_2)$?

Given $f: M_1 \to M_2$ choose a triangulation for M_2

such that any singular value is a vertex

$\to Q \in M_2 : \exists P \in f^{-1}(Q)$

$\to f'(P) = 0$ in some/any coordinate

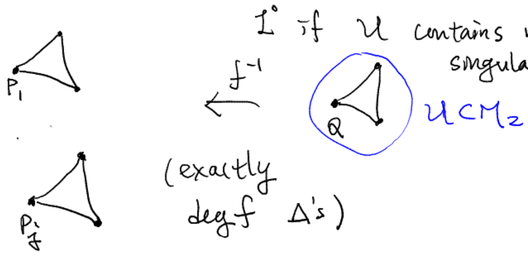
By the inverse function theorem

f^{-1} is a local homeomorphism on $M_2 \setminus \{\text{singular values}\}$

It follows that $f^{-1}(\text{triangulation})$ is almost a triangulation

for M_1, except near $\{P \mid f'(P) = 0\}$

1° if $\mathcal{U}$ contains no singular value

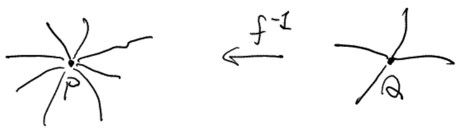


(exactly $\deg f$ Δ's)

2° $P \xrightarrow{f} Q$ $\quad$ $f'(P) = 0$

locally, $z \longmapsto z^{b_f(P)+1} h(z)$ (nowhere zero)

(Say $b_f(P) = 1$)



3° Δ's $\xrightarrow{f^{-1}}$ $\deg f$ (Δ's)

edges $\xrightarrow{f^{-1}}$ $\deg f$ (edges)

vertices $\xrightarrow{f^{-1}}$ $\deg f$ (vertices) $- \sum_{P \in M_1} b_f(P)$ $\quad$ zero, except for finitely many P

$$\Rightarrow \boxed{\chi(M_1) = (\deg f) \cdot \chi(M_2) - \sum_{P \in M_1} b_f(P)}$$

intersection theory on compact Riemann surfaces [FK, § III.1]

On a compact Riemann surface M, $H_1(M)$ consists of the equivalent class of boundaryless curves.

$\to [\sum n_j \gamma_j]$ $n_j \in \mathbb{Z}$, γ_j : C^0, piecewise C^∞, directed curves

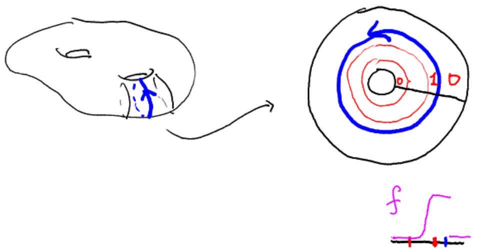
$[\sum n_j \gamma_j] = [\sum \tilde{n}_j \tilde{\gamma}_j]$ if they differ by ∂(some Δ's)

- $H_1(M) \cong \mathbb{Z}^{2g}$ g: genus of M
- they are the obstructions for a closed 1-form to being exact $(\alpha : d\alpha = 0 \Rightarrow \alpha \overset{?}{=} df)$

There is a natural bilinear pairing on $H_1(M)$, which is the "intersection".

recall c : C^0, piecewise C^∞, directed loop

(Poincaré dual) $\Rightarrow \exists$ annulus-type neighborhood



f : NOT C^∞ on M but df is a C^∞ 1-form ("delta" at c)

$$\iint_M \omega \wedge \eta_c = -\int_c \omega \quad \text{for any closed 1-form } \omega$$

Now, for any two cycles on M, define $a \cdot b = \iint_M \eta_a \wedge \eta_b$

$a : \partial a = 0$
formal sum of directed loops — define by linearity

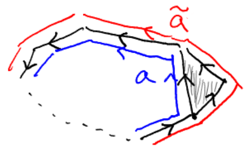
prop • descends to $H_1(M) \times H_1(M) \to \mathbb{Z}$ [1°] $\leftarrow$ algebraic topology: cup product + contraction with the fundamental class

which is [2°] skew-symmetric and [3°] bilinear

$a \cdot b = -b \cdot a$

Moreover, [4°] $a \cdot b$ "counts" the interesection numbers $\leftarrow$ differential topology: transversality

sketch of the proof : 1° $[a] = [\tilde{a}] \in H_1(M)$



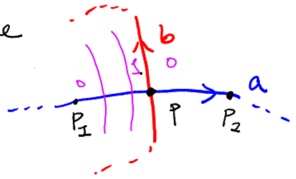
$$\Rightarrow a \cdot b = \iint \eta_a \wedge \eta_b = -\iint \eta_b \wedge \eta_a = \int_a \eta_b$$

$$\tilde{a} \cdot b = \iint \eta_a \wedge \eta_b = \int_{\tilde{a}} \eta_b$$

$$\Rightarrow a \cdot b - \tilde{a} \cdot b = \int_{\partial \Delta} \eta_b = \int_\Delta d\eta_b = 0$$

2° & 3° : easy to see

4° assume locally looks like



nonzero on a nbd of P; near P

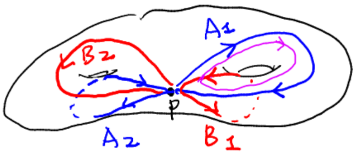
$$\iint \eta_a \wedge \eta_b = \int_a \eta_b = \int_a df = \int_{\overrightarrow{P_1P}} df + \int_{\overrightarrow{PP_2}} df$$

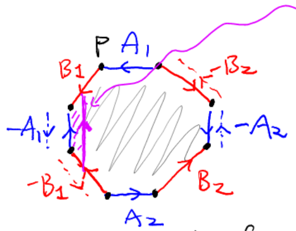
$$= \left(\lim_{Q \to P^-} f(Q) - f(P_1)\right) + \left(f(P_2) - \lim_{Q \to P^+} f(Q)\right) = 1$$

#

Canonical homology basis.

$H_1(M) \cong \mathbb{Z}^{2g}$





representing the same cycles as A_1 in $H_1(M)$

$\Rightarrow A_1 \cdot B_1 = 1$

$A_1 \cdot A_2 = 0 = A_1 \cdot B_2$

the normal form : $4g$-polygon

In general, $A_j \cdot B_k = \delta_{jk} = -B_k \cdot A_j$

$$A_j \cdot A_k = 0 = B_j \cdot B_k$$

$$\Rightarrow \begin{array}{c|cc} & A_k & B_k \\ \hline A_j & A_j \cdot A_k & A_j \cdot B_k \\ B_j & B_j \cdot A_k & B_j \cdot B_k \end{array} = \begin{bmatrix} 0 & \mathbb{I}_{g\times g} \\ -\mathbb{I}_{g\times g} & 0 \end{bmatrix}$$

<u>defn</u> any (integral) basis over $\mathbb{Z}$ for $H_1(\mathbb{Z}) \cong \mathbb{Z}^{2g}$: $\{\mathcal{N}_1, \cdots, \mathcal{N}_{2g}\}$ is called a canonical homology basis

if $[J]_{jk} = \mathcal{N}_j \cdot \mathcal{N}_k = \begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{bmatrix}$ $\quad j,k \in \{1,\cdots,2g\}$

<u>not an example</u> $g=2$. $\mathcal{N}_1 = A_1$, $\mathcal{N}_2 = A_1 + A_2$, $\mathcal{N}_3 = B_1$, $\mathcal{N}_4 = B_2$

$$\Rightarrow J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$