§2 analysis aspect : Wely's lemma , existence of meromorphic functions

Inner product between 1-forms [FK, §I.4]

$$(p\,dx+q\,dy)\wedge *(\bar p\,dx+\bar q\,dy) = (|p|^2+|q|^2)\,dx\wedge dy$$

$$(u\,dz+v\,d\bar z)\wedge *(\bar u\,d\bar z+\bar v\,dz) = (u\,dz+v\,d\bar z)\wedge(+i\bar u\,d\bar z - i\bar v\,dz)$$
$$= (|u|^2+|v|^2)\,\underbrace{i\,dz\wedge d\bar z}_{=2\,dx\wedge dy}$$

rmk $p = u+v \quad q = i(u-v)$

Fix $D\subset M$: open & connected. $\begin{cases}\partial D : \text{continuous \& piecewise smooth curves}\\ \bar D : \text{compact}\end{cases}$

$\omega = u\,dz + v\,d\bar z$ define $\|\omega\|_D^2 = \iint_D \omega\wedge *\bar\omega$

$(\omega_1,\omega_2)_D = \iint_D \omega_1\wedge *\overline{\omega_2}$

enlarge the class

- eventually, care about the smooth ones,
- for some by parts formula, only require u,v to be C^1 or C^2
- for some technical treatment in analysis.

$$L^2(D) = \{u,v : \text{measurable} \mid \iint_D \omega\wedge *\bar\omega < \infty\}$$

key point to introduce $L^2(D)$ ← complete vector space
i.e. Hilbert space

prop i) $*^2 = -\mathrm{Id}$, $*$: isometry

ii) $(d\varphi, *\alpha)_D = \iint_D \varphi\, d\bar\alpha - \int_{\partial D}\varphi\bar\alpha$ (φ: C^1 function, α: C^1 1-form defined on a nbd of $\bar D$)

iii) $(d\varphi, d\psi)_D = -\iint_D \varphi\,\Delta\bar\psi + \int_{\partial D}\varphi * d\bar\psi$ (φ,ψ : C^2 function)

$(d\varphi, *d\psi)_D = -\int_{\partial D}\varphi\, d\bar\psi$

$\Delta\psi := -2i\,\bar\partial\partial\psi = d*d\psi = (\psi_{xx}+\psi_{yy})\,dx\wedge dy$

iv) f: holomorphic function , ω: holomorphic differential (defined on a nbd of $\bar D$)

$$\iint_D df\wedge\bar\omega = 2\int_{\partial D}(\mathrm{Re} f)\,\bar\omega = 2i\int_{\partial D}(\mathrm{Im} f)\,\bar\omega$$

pf: i) is direct

ii) $(d\varphi, *\alpha)_D = -\iint_D d\varphi\wedge\bar\alpha = \iint_D \varphi\, d\bar\alpha - \iint_D d(\varphi\wedge\bar\alpha) = \iint_D\varphi\, d\bar\alpha - \int_{\partial D}\varphi\bar\alpha$

but $d(\underbrace{\varphi}_{1}\bar\alpha) = \underbrace{d\varphi}_{1}\wedge\underbrace{\bar\alpha}_{1} + \varphi\,\underbrace{d\bar\alpha}_{2}$ ← $\boxed{\text{check}}$

iii) $(d\varphi, d\psi)_D = (d\varphi, -*(*d\psi))_D = -\iint_D \varphi\,\underbrace{d*d\bar\psi}_{\Delta\bar\psi} + \int_{\partial D}\varphi * d\bar\psi$

$\alpha = -*d\psi$

$(d\varphi, *d\psi)_D = +\iint_D \varphi\, \cancel{d(d\bar\psi)}^{\,0} - \int_{\partial D}\varphi\, d\bar\psi$

iv) f: holomorphic $\Rightarrow \bar\partial f = 0$ $\quad \omega = \alpha(z)\,dz$, α holo $\quad dz\wedge dz = 0$

$d(f\omega) = (df)\wedge\omega + f\,\cancel{d\omega} = \cancel{\partial f\wedge\omega} = 0$

$$\Rightarrow \iint_D d(fw) = 0 = \int_{\partial D} fw \Rightarrow \int_{\partial D} (\mathrm{Re} f) w = -i \int_{\partial D} (\mathrm{Im} f)\, \omega$$

$$\iint_D df \wedge \bar{\omega} = (df, -*\omega)_D \underset{\text{by ii)}}{=} -\iint_D df \wedge \cancel{d\bar{\omega}}^{\,0} + \int_{\partial D} f \bar{\omega} \quad \cdots \qquad \#$$

basic facts about Hilbert spaces [FK, § II. 1]

goal construct meromorphic fuctions on M

if there is a holomorphic differential $\rightsquigarrow$ multiply $\rightsquigarrow$ a meromorphic differential

or, if we have two different meromorphic differential, their "quotient" is a meromorphic function

the construction of meromorphic differential can be done by using harmonic 1-forms.

$\rightarrow$ construct harmonic 1-forms ($\perp_{L^2}$ d(functions), also $*$d(functions))

orthogonal complement in $L^2 \overset{?}{\rightsquigarrow}$ (smooth) harmonic

- $(H, (\,,\,))$: Hilbert space
  $\overset{\cup}{F}$ (vector) subspace $\Rightarrow F^\perp$ is a closed subspace
- if F is closed $\Rightarrow H = F \oplus F^\perp$
  and the projection $P: H = F \oplus F^\perp \to F$ is a bounded linear operator (of norm 1)
- Riesz representation theorem. $H^* = H$
  namely, any bounded linear functional is $\langle x, - \rangle$ for some $x \in H$

Weyl's lemma [FK. § II.2]

thm D: unit disk. φ: square integrable measurable function

φ is harmonic if and only if $\iint_D \varphi \Delta \eta = 0$

(regularity theorem) $\forall \eta$: smooth, with compact support $\subset D$

Pf: $\Rightarrow$ say $\mathrm{supp}\, \eta \subset D_r \quad r<1$

$$\iint_{D_r} \varphi \Delta \eta = \iint_{D_r} \varphi\, d*d\eta = \iint_{D_r} d\varphi \wedge *d\eta - \cancel{\int_{\partial D_r} \varphi * d\eta}^{\,0} \text{ support}$$

$$= -\iint_{D_r} *d\varphi \wedge d\eta = \iint_{D_r} d\eta \wedge *d\varphi = \iint_{D_r} \eta\, \cancel{d*d\varphi}^{\,0} \quad \varphi\text{: harmonic}$$

$\Leftarrow$ some ideas
- try to get information on $\iint_D \varphi \mu \, dx \wedge dy$
  $\rightsquigarrow \Delta \eta = \mu\, dx \wedge dy \rightsquigarrow \eta = \iint_D G(z, \zeta)\, \mu(\zeta) \dfrac{d\zeta \wedge d\bar{\zeta}}{-2i}$
  G: green kernel of $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, ΔG = delta distribution
  $G \sim \log|z-\zeta|$
- if φ is really harmonic $\Rightarrow$ mean value property ?

0° Green's 2nd identity & Green's representation formula

u, v: real valued smooth function

$$\iint_\Omega u \Delta v - v \Delta u = \int_{\partial\Omega} u \frac{\partial v}{\partial n} - u \frac{\partial u}{\partial n}\, ds$$

arc length — (Green's 2nd identity)

unit outer normal

u: compact support on $\mathbb{R}^2$

Consider $\eta(z) = \iint_{\mathbb{R}^2} \frac{\log|z-\zeta|}{2\pi} u(\zeta) \frac{d\zeta \wedge d\bar\zeta}{-2i}$ $\qquad \xi = \zeta - z$

$$= \iint_{\mathbb{R}^2} \frac{\log|\xi|}{2\pi} u(z+\xi) \frac{d\xi \wedge d\bar\xi}{-2i}$$

differentiation hits here $\Rightarrow$ $\eta(z)$ is smooth

(temporarily, set $\Delta\eta$ to be $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\eta$, but not $(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2}) dx\wedge dy$)

$$\Delta_z\eta(z) = \iint_{\mathbb{R}^2} \frac{\log|\xi|}{2\pi} \Delta_z u(z+\xi) \frac{d\xi\wedge d\bar\xi}{-2i}$$

$$= \iint_{\mathbb{R}^2} \frac{\log|z-\zeta|}{2\pi} \Delta_\zeta u(\zeta) \frac{d\zeta\wedge d\bar\zeta}{-2i}$$

$$= \lim_{\rho\to 0} \iint_{\mathbb{R}^2\setminus D_r(z)} \frac{\log|z-\zeta|}{2\pi} \Delta_\zeta u(\zeta) \frac{d\zeta\wedge d\bar\zeta}{-2i} = u(z)$$

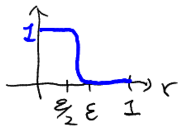
$\left(\begin{array}{l} s+t=\lambda \\ u(s+t) = u(\lambda) \\ \frac{\partial u}{\partial s}(s+t) = u'(s+t) = u'(\lambda) \\ \quad = \frac{\partial}{\partial r}u(\lambda) \end{array} \right)$

$$\iint_{\mathbb{R}^2\setminus D_r(z)} \frac{\log|z-\zeta|}{2\pi} \Delta_\zeta u(\zeta) = \frac{1}{2\pi}\int_{\partial D_r(z)} \left(u(\zeta) \frac{\partial \log|z-\zeta|}{\partial n} - \log|z-\zeta| \frac{\partial u}{\partial n}\right) ds$$

$\Delta_\zeta = 0$ $\boxed{\text{check}}$ $\qquad \frac{\partial \log|z-\zeta|}{\partial n} = \frac{1}{r}$ $\qquad \log|z-\zeta| = \log r$ $\qquad u\in C^\infty \Rightarrow |\frac{\partial u}{\partial n}|$ bdd near $\zeta = z$ $\qquad ds = r\,d\theta$

<u>But</u> $\eta(z)$ is not of compact support.

1° Fix $\varepsilon > 0$ (small), let $\rho(r) =$



For any $\mu(z)$: C^∞ on D with supp $\subset D_{1-\varepsilon}$

Consider $\tilde\eta(z) = \iint_D \rho(|z-\zeta|) \frac{\log|z-\zeta|}{2\pi} \mu(\zeta) \frac{d\zeta\wedge d\bar\zeta}{-2i}$

{ supp $\tilde\eta \subset D_{1-\varepsilon/2}$
replace D by $\mathbb{C}$

$$= \iint_{\mathbb{C}} \rho(|z-\zeta|) \frac{\log|z-\zeta|}{2\pi} \mu(\zeta) \frac{d\zeta\wedge d\bar\zeta}{-2i}$$

$$= \iint_{\mathbb{C}} \frac{\log|z-\zeta|}{2\pi} \mu(\zeta) \frac{d\zeta\wedge d\bar\zeta}{-2i} + \iint_{\mathbb{C}} (\rho(|z-\zeta|)-1) \frac{\log|z-\zeta|}{2\pi} \mu(\zeta) \frac{d\zeta\wedge d\bar\zeta}{-2i}$$

$$\Delta_z\tilde\eta(z) = \mu(z) + \iint_{\mathbb{C}} \underbrace{\Delta_z\left((\rho(|z-\zeta|)-1) \frac{\log|z-\zeta|}{2\pi}\right)}_{\text{vanishes when } |z-\zeta|>\varepsilon \text{ and } |z-\zeta|<\varepsilon/2} \mu(\zeta) \frac{d\zeta\wedge d\bar\zeta}{-2i}$$

similar proof as the Green's representation formula

denote it by $\gamma(z,\zeta)$

2° $\iint_D \varphi(z)\,\Delta_z\tilde\eta(z) = 0$

$$= \iint_D \varphi(z)\mu(z) \frac{dz\wedge d\bar z}{-2i} + \iint_D \varphi(z)\left(\iint_D \gamma(z,\zeta)\mu(\zeta) \frac{d\zeta\wedge d\bar\zeta}{-2i}\right) \frac{dz\wedge d\bar z}{-2i}$$

unit disk in the z-plane $\qquad$ unit disk in the ζ-plane

$$\Rightarrow \iint_D \varphi(\zeta)\mu(\zeta) \frac{d\zeta\wedge d\bar\zeta}{-2i} + \iint_D \mu(\zeta)\left(\iint_D \varphi(z)\gamma(z,\zeta) \frac{dz\wedge d\bar z}{-2i}\right) \frac{d\zeta\wedge d\bar\zeta}{-2i} = 0$$

by Fubini

this is true for all μ $\Rightarrow$ $\varphi(\zeta) = -\iint_D \varphi(z)\gamma(z,\zeta) \frac{dz\wedge d\bar z}{-2i}$ $\quad \forall |\zeta| < 1-2\varepsilon$

smooth in ζ

$\Rightarrow$ $\varphi(\zeta)$ is smooth on $D_{1-2\varepsilon}$ $\forall\varepsilon$

※

harmonic differential [FK, §II.3]

<u>recall</u> a differential ω is said to be harmonic if $d\omega = 0 = d*\omega$

if and only if ω is locally d(harmonic function)

<u>observation</u> f: smooth function on M with compact support

→ do not require compactness at this moment

ω: harmonic
$$(\omega, df)_M = (*\omega, *df) = -\iint_M *\omega \wedge d\bar{f} = -\iint_M (d*\omega)\bar{f} = 0$$
$$(\omega, *df)_M = -\iint_M \omega \wedge d\bar{f} = -\iint_M (d\omega)\bar{f} = 0$$

$\Rightarrow$ harmonic differential $\perp_{L^2(M)}$ df & $*df$

<u>defn</u> $E = L^2(M)$-closure of $\{df \mid f: \mathcal{C}^\infty, \text{compact supp}\}$

$E^* = *E$ $\left\{ \begin{array}{l} *: \text{also defined on } L^2(M) \\ E^* = L^2(M)\text{-closure of } \{*df \mid f: \mathcal{C}^\infty \text{ compact supp}\} \end{array} \right.$

$\Rightarrow L^2(M) = E \oplus E^\perp$
$= E^* \oplus E^{*\perp}$

By the basic properties of Hilbert spaces
$E^\perp = \{\omega \in L^2(M) \mid (\omega, df) = 0 \ \forall f: \mathcal{C}^\infty, \text{cpt-supp}\}$
$(E^*)^\perp = \{\omega \in L^2(M) \mid (\omega, *df) = 0 \ \forall f: \mathcal{C}^\infty, \text{cpt-supp}\}$

relation between these two decomposition ?

characterization of $E^\perp$ & $E^{*\perp}$:

<u>prop</u> $\alpha \in L^2(M)$ of class $\mathcal{C}^1$. Then, $\alpha \in E^\perp$ if and only if $d*\alpha = 0$

pf: $\Leftarrow$ easy

$\Rightarrow$ $\alpha \in E^\perp$: $(\alpha, df) = 0 \quad \forall f: \mathcal{C}^\infty$, cpt-supp

$= -\iint_M (d*\alpha)\bar{f}$ (as the observation) ✕

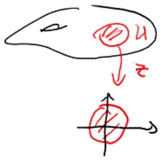
It follows that $E^* \subset E^\perp$. Similarly, $E \subset E^{*\perp}$

$*df \Rightarrow d*(*df) = d^2\bar{f} = 0$

<u>Hence</u>, if we let $H = E^\perp \cap (E^*)^\perp$

then, $L^2(M) = H \oplus E \oplus E^*$ (compare with the observation)

What is H? $\omega \in H \Rightarrow (\omega, df) = 0 = (\omega, d*f) \quad \forall f: \mathcal{C}^\infty$, cpt-supp



examine it locally (on some nbd of any point)

$\omega = p\,dx + q\,dy \qquad df = f_x\,dx + f_y\,dy$
$\qquad\qquad\qquad\qquad\quad d*f = -f_y\,dx + f_x\,dy$

$$\iint (p\bar{f}_y - q\bar{f}_x)\,dx\wedge dy = 0 = \iint (p\bar{f}_x + q\bar{f}_y)\,dx\wedge dy$$

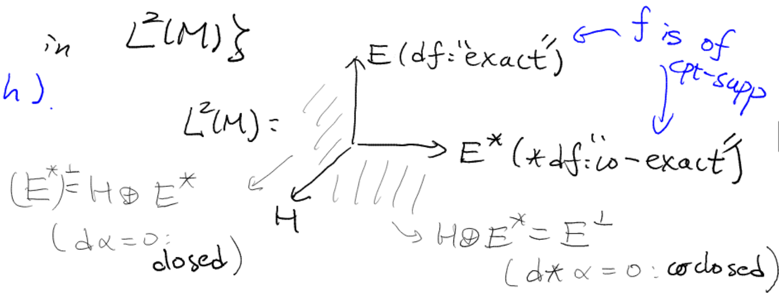
g: $\mathcal{C}^\infty$, cpt-supp, consider $f = \bar{g}_x, \bar{g}_y$

$$\Rightarrow \iint p(g_{xx} + g_{yy})\,dx\wedge dy = 0 = \iint q(g_{xx} + g_{yy})$$

By Weyl's lemma, ω is smooth & harmonic on this nbd

<u>thm</u> $H = \{\text{harmonic differential in } L^2(M)\}$

(Weyl $\Rightarrow$ smooth).



<u>Caveat:</u> when M is not compact, E may not consist of all exact forms (f: compact support)

for instance, $M = D$, $\exists f$: harmonic on D

$\Rightarrow df \in H \perp E$

a criterion for exactness

($\alpha = df$ for some $f \in \mathcal{C}^2$)

<u>prop</u> $\alpha \in L^2(M)$ of $\mathcal{C}^1$ and closed. Then, α is exact if and only if $(\alpha, \beta) = 0 \quad \forall \beta: \mathcal{C}^\infty$ differential $\begin{cases} \text{cpt-supp} \\ d*\beta = 0 \end{cases}$

pf: $\Rightarrow$ $(\alpha,\beta) = \iint_M df \wedge *\bar{\beta} = -\iint_M f\, d*\bar{\beta} = 0$

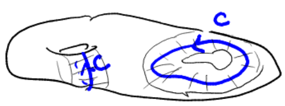
$\Leftarrow$ If $\int_C \alpha = 0 \quad \forall$ closed (directed) curve $C \subset M$, for the well-definedness

we can define $f(q) = \int_\gamma \alpha \quad \gamma$: any curve from p to q

It is straightforward to check that $df = \alpha$

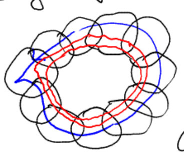
$\left(\begin{array}{l} \alpha = g\,dx + h\,dy \\ \gamma: \text{rectangular path near } q \end{array} \right)$

<u>goal</u> $(\alpha,\beta) = 0 \Rightarrow \int_C \alpha = 0$



C: $\mathcal{C}^0$, piecewise $\mathcal{C}^\infty \overset{\text{(diff)}}{\underset{\text{top}}{\Longrightarrow}} \exists$ nbd of C in M $\cong$ annulus

slightly more detail:



C admits finite coordinate cover.

Construct two $\mathcal{C}^\infty$ curves on the left hand side of C

Choose two $\mathcal{C}^\infty$ curves

choose f on the annulus $\setminus C$

so that $\begin{cases} f = 0 & \Omega_2^- \text{ and } \Omega^+ \\ f = 1 & \Omega_0^- \end{cases}$

$\Rightarrow df$ is a smooth form <u>on M</u> of cpt-supp

Denote it by η_c ← NOT d(smooth function on M)

Easy to check $d\eta_c = 0$ $\boxed{\text{think}}$

Consider $(\alpha, *\eta_c) = -\iint_M \alpha \wedge \eta_c = -\iint_{\Omega^-} \alpha \wedge df$

$0 = d*(*\eta_c)$

$= \iint_{\Omega^-} d(f\alpha) - \iint f\, \cancel{d\alpha}^{\,0}$ (α: closed) $= \int_{\partial \Omega^-} f\alpha = \int_C \alpha$

<u>rmk</u> η_c is usually referred as the Poincaré dual of C ※

On a <u>compact</u> Riemann surface M, exact harmonic differential $\in H \cap E = \{0\}$

$\Rightarrow$ harmonic functions must be constant

$\overset{d}{\searrow}$ harmonic differential $= 0$ $\Uparrow$

To construct a meromorphic function, consider "harmonic functions and differentials with singularity" summary : Hodge decomposition

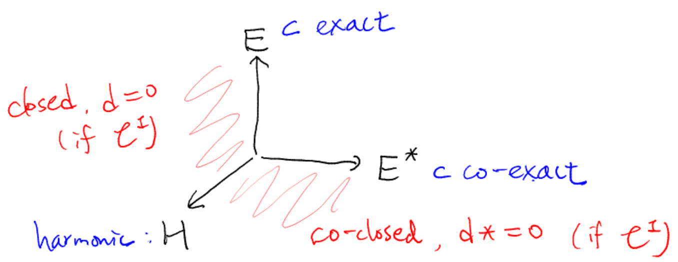
function $\xrightarrow{d}$ differential $\xrightarrow{d}$ 2-form $\qquad E = \overline{\{df\}} \qquad E^* = \overline{\{*df\}}$

$\circlearrowleft *$ $\qquad$ ($\leftarrow$: see the HW) $\qquad f : \mathcal{C}^\infty$, cpt-supp

$E \perp E^*$

$\Rightarrow L^2(M) = H \oplus E \oplus E^*$ $\qquad H =$ harmonic differentials , $d\omega = 0 = d*\omega$

$\omega \overset{\text{locally}}{=} d$(harmonic function)



prop $\omega : \mathcal{C}^1$, $d\omega = 0$ Then, $\omega = df$ if and only if $(\omega, \beta) = 0$

$\forall \beta : \mathcal{C}^\infty$, cpt-supp , $d*\beta = 0$

- works for non-compact M as well
- relies on the construction of the Poincaré dual

C : closed (directed) curve $\subset M \rightsquigarrow \eta_c \quad d\eta_c = 0 \quad \int_M \omega \wedge \eta_c = \int_C \omega$

($\rightsquigarrow$ H is closely related to closed curves on M)

e.g. (detail : Fourier transform ⋯)

on $T^2 = \mathbb{C}/(2\pi\mathbb{Z})^2$ $\qquad$ functions $= \sum_{n,m} a_{n,m} e^{i(nx+my)} \longleftrightarrow \{a_{n,m}\}$ $\qquad \ell^2$

1-form $\overset{\boxed{HW}}{=}$ (function, function) $\longleftrightarrow$ two Fourierier coefficients

$(\{b_{n,m}\}, \{c_{n,m}\})$ $\qquad \ell^2 \oplus \ell^2$

$d : \{a_{n,m}\} \longmapsto (\{n a_{n,m}\}, \{m a_{n,m}\})$

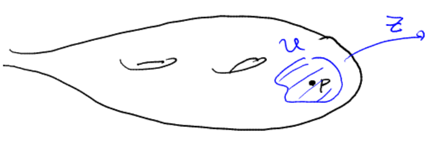
$* : (\{b_{n,m}\}, \{c_{n,m}\}) \longmapsto (\{-c_{n,m}\}, \{b_{n,m}\})$

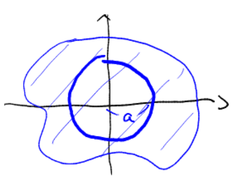
$d : (\{b_{n,m}\}, \{c_{n,m}\}) \longmapsto (\{n c_{n,m} - m b_{n,m}\})$

$\boxed{\text{check}}$

$$\Rightarrow (\{b_{n,m}\}, \{c_{n,m}\}) = \underbrace{(\{b_{0,0}\}, \{c_{0,0}\})}_{H} + \underbrace{d\left\{\frac{n b_{n,m} + m c_{n,m}}{n^2+m^2}\right\}_{(n,m)\neq(0,0)}}_{E} + \underbrace{*d\left\{\frac{-m b_{n,m} + n c_{n,m}}{n^2+m^2}\right\}_{(n,m)\neq(0,0)}}_{E^*}$$

constructing harmonic function with singularity

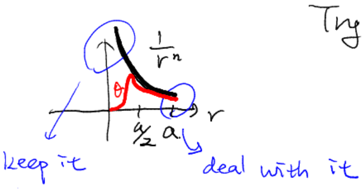




assume $z(p) = 0 \qquad D_a \subset z(\mathcal{U})$

$\Rightarrow \frac{1}{z^n}$ is a locally defined function on M (only on $\mathcal{U}$)

Try to extend it over M.



$f = \frac{1}{z^n} + ?$ (tool : Hodge decomposition)

$\theta : \mathcal{C}^\infty$ on D_a

$\theta = \frac{1}{z^n} + (\cdots)$ on $\{z \mid \frac{a}{2} \le z < a\}$

$\nwarrow$ some modification, explain later

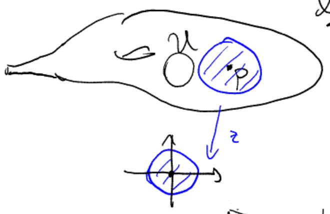
$\Rightarrow d\theta \in L^2(D_a) \underset{\substack{\text{extend} \\ \text{by } 0}}{\subset} L^2(M)$

Hodge decomposition of $d\theta$?

$\downarrow$

not $\mathcal{C}^\infty$, cpt-supp $\Rightarrow d\theta$ may not be in E

try to analyze its E-component



Lemma (extension lemma) θ : smooth on $\{|z|<a\}$. harmonic on $\{\frac{a}{2}<|z|<a\}$

$*d\theta = 0$ on $\{|z|=a\}$ (in the sense of $\lim_{\rho\to a^-} *d\theta|_{\{|z|=\rho\}} = 0$)

$d\theta \in L^2(\{|z|<a\})$

Regard $d\theta$ as an element in $L^2(M)$, $d\theta = \alpha + \omega \in E \oplus (E^* \oplus H)$

Then, $\alpha = df$, f: smooth, harmonic on $M \setminus \{|z| \le a/2\}$

and $f - \theta$ is harmonic on $\{|z|<a\}$

pf: 0° $\mathcal{U}$: a coordinate disk with $w = u + iv$ $\quad \alpha|_{\mathcal{U}} = p\,du + q\,dv$

$\rho : \mathcal{C}^\infty$- real-valued, $\operatorname{supp}\rho \subset \mathcal{U}$

$\Rightarrow \iint p\Delta\rho = ?$ relate it to θ $\quad E \leftarrow$ orthogonal $\rightarrow E^*$

$\iint p(\rho_{uu}+\rho_{vv})\,du\wedge dv = (\alpha, (\rho_{uu}+\rho_{vv})\,du) = (\alpha, d(\frac{\partial\rho}{\partial u}) - *d(\frac{\partial\rho}{\partial v}))$

By the similar trick for proving $H = \{$harmonic differentials$\}$,

write $\rho_{uu}\,du + \rho_{vv}\,du = d(\rho_u) - \rho_{uv}\,dv + \rho_{vv}\,du = d(\frac{\partial\rho}{\partial u}) - *d(\frac{\partial\rho}{\partial v})$

$= (\underset{\substack{\cap\\ E}}{\alpha}, \underset{\substack{\cap\\ E}}{d(\frac{\partial\rho}{\partial u})}) = (\alpha+\omega, d(\frac{\partial\rho}{\partial u})) = (d\theta, d(\frac{\partial\rho}{\partial u}))$

Similarly, $\iint q\Delta\rho = (d\theta, d(\frac{\partial\rho}{\partial v}))$

1° if $\mathcal{U} \subset M \setminus \{|z| \le a/2\}$

$\iint_{\mathcal{U}} p\Delta\rho = (d\theta, \underbrace{d(\frac{\partial\rho}{\partial u})}_{\varphi})_M = \cancelto{0}{(d\theta, d\varphi)_{D_{a/2}}} + (d\theta, d\varphi)_{D_a \setminus D_{a/2}} + \cancelto{0}{(d\theta, d\varphi)_{M\setminus D_a}}$

$D_a\setminus D_{a/2} \leadsto D_r \setminus D_{a/2}$, $r\to a^-$ $\qquad M\setminus D_a \leadsto M\setminus D_r$, $r \to a^+$

$(d\theta, d\varphi)_{D_a\setminus D_{a/2}} = \iint_{D_a\setminus D_{a/2}} d\theta \wedge * d\varphi = -\iint_{D_a\setminus D_{a/2}} *d\theta \wedge d\varphi$

$= \iint_{D_a\setminus D_{a/2}} d((*d\theta)\varphi) - \cancelto{0}{(d*d\theta)}\varphi$ $\quad \theta$: harmonic

$= \int_{\partial D_a} (*d\theta)\varphi - \int_{\partial D_{a/2}} (*d\theta)\cancelto{0}{\varphi}$ $\quad$ support

$\lim_{r\to a^-} *d\theta|_{\partial D_r} = 0$

$d\theta = \frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy$

$*d\theta = -\frac{\partial\theta}{\partial y}dx + \frac{\partial\theta}{\partial x}dy$

on ∂D_r $\quad x = r\cos t$, $y = r\sin t$

$dx = -r\sin t\,dt$, $dy = r\cos t\,dt$

$*d\theta|_{\partial D_r} = (x\frac{\partial\theta}{\partial x} + y\frac{\partial\theta}{\partial y})|_{\partial D_r}\,dt$

$\overset{\text{chain rule}}{=} (r\frac{\partial\theta}{\partial r})|_{\partial D_r}\,dt$

$= (\frac{\partial\theta}{\partial n})\,r\,dt = \frac{\partial\theta}{\partial n}\,ds$

$\Rightarrow \iint_{\mathcal{U}} p\Delta\rho = 0$, Similarly $\iint_{\mathcal{U}} q\Delta\rho = 0$

$\forall \rho : \mathcal{C}^\infty$. $\operatorname{supp} \subset \mathcal{U}$

By Weyl's lemma, $p, q : \mathcal{C}^\infty$ & harmonic on $\mathcal{U}$

$\Rightarrow \alpha : \mathcal{C}^\infty \underset{\text{Hodge on } \mathcal{U}}{\Rightarrow}$ closed on $\mathcal{U} \subset M \setminus \{|z| < a/2\}$

2° Study α on $D_a = \{|z|<a\}$

Take any $\rho : \mathcal{C}^\infty$. $\operatorname{supp} \subset D_a = \{|z|<a\}$

$\iint_{D_a} p\Delta\rho = (d\theta, d(\frac{\partial\rho}{\partial x})) = \iint_{D_a} (\theta_x \rho_{xx} + \theta_y \rho_{xy})\,dx\wedge dy$

$0 = (d\theta, -*d(\frac{\partial\rho}{\partial y})) = \iint_{D_a} (\theta_x \rho_{yy} - \theta_y \rho_{xy})\,dx\wedge dy$

$\Rightarrow \iint_{D_a} (p - \theta_x)\Delta\rho = 0 \quad \Rightarrow \quad p - \theta_x$ is harmonic on D_a

Similarly, $q - \theta_y$ is harmonic on D_a

$\Rightarrow \alpha$ is C^∞ on $D_a \Rightarrow$ closed on D_a

3° It follows that α is C^∞ on $M \setminus \overline{D}_{a/2} \cup D_a = M$

Since $\alpha \perp_{L^2} \{ C^\infty$ 1-form, co-closed $(*d = 0)$, cpt-supp$\}$, $\alpha = df$

$f : C^\infty$

harmonicity of f ? $\Leftrightarrow$ harmonicity of α

- on $M \setminus D_{a/2}$. $\alpha|_{M \setminus D_{a/2}} \perp E^*$ (over $M \setminus D_{a/2}$)

$\rho : C^\infty$, supp $\subset M \setminus D_{a/2}$. $(\alpha, d\rho)_{M \setminus D_{a/2}} = (\alpha, d\rho)_M = (\alpha + \omega, d\rho)_M$

$= (d\theta, d\rho)_M$

$= (d\theta, d\rho)_{D_a \setminus D_{a/2}} = 0$

$\downarrow$ same argument as in 1°

- on D_a, $d(\theta - f) = d\theta - \alpha = \omega \in E^*$

$\Rightarrow \omega : C^\infty$ on D_a

Similarly, $(d\rho, \omega) = 0 \quad \forall \rho : C^\infty$. supp $\rho \subset D_a$

$= \iint d\rho \wedge * \overline{\omega}$

$= -\iint \rho (d * \overline{\omega})$

$\Rightarrow d * \omega = 0 \Rightarrow d * d(\theta - f) = 0$ ※

Now, consider $h = \begin{cases} \frac{1}{z^n} + \frac{z^n}{a^{2n}} & \text{if } |z| < a \\ 0 & \text{o.w.} \end{cases} \rightsquigarrow \theta = \begin{cases} h(z) & |z| \geq a/2 \\ C^\infty & |z| < a \end{cases}$

also harmonic, makes $*dh|_{\partial D_a} = 0 = *d\theta|_{\partial D_a}$

$\Rightarrow$ By applying the extension lemma to θ,

we get $f : C^\infty$ on M. harmonic on $M \setminus \overline{D}_{a/2}$

Consider $u = h - \theta + f$

on $M \setminus \overline{D}_{a/2}$, $u = \cancel{h - \theta} + f$: harmonic

on D_a. $u = h + (f - \theta)$: harmonic summary: criterions for closedness, coclosedness, exactness

$\alpha \in \mathcal{C}^1$ i) $d\alpha|_{\mathcal{U}} = 0 \iff \iint (d\alpha) f = 0 \quad \forall f: \mathcal{C}^\infty$, cpt-supp, $\operatorname{supp} f \subset \mathcal{U}$

$\iff \iint \alpha \wedge df = 0 \iff (\alpha, *df) = 0$

ii) $d{*}\alpha|_{\mathcal{U}} = 0 \iff (*\alpha, *df) = 0 \iff (\alpha, df) = 0$

easy to check

hard part. $\alpha \in \mathcal{C}^1$ (regularity)

iii) $\alpha = df$ then $(\alpha, \beta) = (df, \beta) = \iint df \wedge *\bar{\beta} = -\iint f \, d{*}\bar{\beta}$

also true by Poincaré dual construction

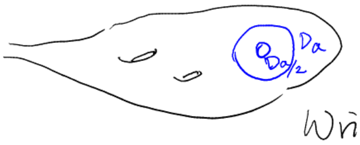
supp β: cpt

$= 0$ if $d{*}\beta = 0$

In particular, if $\alpha \in \overline{E}$ and $\mathcal{C}^\infty \Rightarrow \alpha = df$

(Since $E = \{dg\}$ $g: \mathcal{C}^\infty$, cpt-supp, $\Rightarrow g \perp \beta \Rightarrow \overline{E} \perp \beta$ $\forall \beta$ as above)

extension lemma



θ: $\mathcal{C}^\infty$ on D_a, 0 on $M \setminus \overline{D_a}$, harmonic on $D_a \setminus \overline{D_{a/2}}$

$*d\theta|_{\partial D_a} = 0$, $d\theta \in L^2(M)$

Write $d\theta = \underset{\overline{E}}{\alpha} + \underset{H \oplus E^*}{\omega}$ Then $\begin{cases} \alpha = df & f: \mathcal{C}^\infty \ (\Leftarrow \boxed{\alpha: \mathcal{C}^\infty}) \\ f: \text{harmonic on } M \setminus \overline{D_{a/2}} \\ f - \theta: \text{harmonic on } D_a \end{cases}$

key
- $\alpha \in \mathcal{C}^\infty$ by Weyl's lemma
- $d{*}\alpha|_{M \setminus \overline{D_{a/2}}} \equiv 0$ $\quad (\alpha, d\rho) \overset{d\rho \in E}{=} (d\theta, d\rho) \overset{\text{similar as last time}}{=} \int_{\partial D_a} \rho * d\theta = 0$ $\quad$ supp $\rho \subset M \setminus \overline{D_{a/2}}$
- $d{*}(d\theta - \alpha)|_{D_a} \equiv 0$ $\quad (d\theta - \alpha, d\rho) = (\underset{H \oplus E^*}{\omega}, \underset{E}{d\rho}) = 0$ $\quad$ supp $\rho \subset D_a$

think more about the domain

the condition $*dh = 0$ on ∂D_a

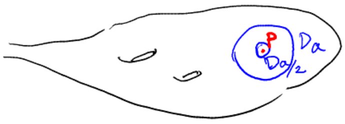
$*dh|_{\partial D_a} = \frac{\partial h}{\partial n}\Big|_{\partial D_a} ds \left(= \frac{\partial h}{\partial r}\Big|_{r=a} \cdot r\,dt \right)$ $\quad z = re^{it}$

How to make it zero? Consider the reflection $z \mapsto \frac{a^2}{\bar{z}}$

Given $f(z)$, consider $f(z) + f\left(\frac{a^2}{\bar{z}}\right)$ $\quad re^{it} \mapsto \frac{a^2}{r} e^{it}$

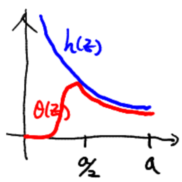
identity on ∂D_a

$\frac{\partial}{\partial r}$ at $r = a$ $\quad \frac{\partial f}{\partial z} e^{it} + \frac{\partial f}{\partial \bar{z}} e^{-it} - \frac{\partial f}{\partial z} \frac{a^2}{a^2} e^{it} - \frac{\partial f}{\partial \bar{z}} \frac{a^2}{a^2} e^{-it} = 0$



Consider $h(z) = \begin{cases} \frac{1}{z^n} + \frac{\bar{z}^n}{a^{2n}} & z \in D_a \\ 0 & \text{otherwise} \end{cases}$

rough picture



$\theta(z) = \begin{cases} h(z) & z \in M \setminus \overline{D_{a/2}} \\ \mathcal{C}^\infty & z \in D_a \end{cases}$

$\rightsquigarrow f$: the function given by the extension lemma

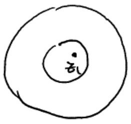
$\Rightarrow u = h - \theta + f$: $\mathcal{C}^\infty$ and harmonic on $M \setminus p$

examine it on $M \setminus \overline{D_{a/2}}$ and D_a

near p, $u = \frac{1}{z^n} + (\text{smooth})$

What about $\log|z|$? ($\log|z|$: harmonic on $\mathbb{C}\setminus\{0\}$)

$\log|z| + \log\left|\frac{a^2}{\bar z}\right| = \log a^2$ = constant : not useful

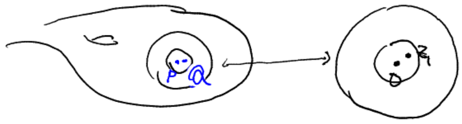


choose any $z_1 \in D_{a/2}\setminus\{0\}$

$$\log|z-z_1| + \log\left|\frac{a^2}{\bar z} - z_1\right| = \log\left|\frac{1}{z}(z-z_1)\left(z-\frac{a^2}{\bar z_1}\right)\right| + \log|z_1|$$

$$\overline{\left(\frac{a^2}{\bar z}-z_1\right)} = \frac{a^2}{z} - \bar z_1 = -\bar z_1 \, \frac{1}{z}\left(z - \frac{a^2}{\bar z_1}\right)$$

$$\Rightarrow h(z) = \begin{cases} \log\left|\frac{1}{z}(z-z_1)\left(z-\frac{a^2}{\bar z_1}\right)\right| & z \in D_a \\ 0 & \text{otherwise} \end{cases}$$



By the same construction as above, $u = h - \theta + f$ is harmonic on $M\setminus\{P,Q\}$

and $\begin{cases} u = -\log|z| + (C^\infty) & \text{near } P \\ u = \log|w| + (C^\infty) & \text{near } Q \end{cases}$

we will see why it is the case momentarily

↳ coordinate on a neighborhood of Q, such that $w(Q)=0$

in fact, $w = z - z_1$ would do the job

prop Given any $P, Q \in M$,

$\exists\, u$: harmonic on $M\setminus\{P,Q\}$,

and $u = \begin{cases} -\log|z| + (C^\infty) & \text{on a neighborhood of } P \\ \log|w| + (C^\infty) & \text{on a neighborhood of } Q \end{cases}$

pf: as above

meromorphic differential & meromorphic function [FK, §II.5]

defn a meromorphic differential (abelian differential) is

$f(z)\,dz$ on each coordinate chart (of course, must obey the transition rule)

↳ meromorphic function

u: harmonic function v = conjugate harmonic

$\Rightarrow u + iv$ = holomorphic $\Rightarrow du + i\,dv = du + i * du$: holomorphic differential

In general, given a harmonic differential α,

$\alpha + i * \alpha$ is a holomorphic differential

$$\left(\begin{array}{ll} \alpha = g\,dz + h\,d\bar z & \text{harmonic} \Leftrightarrow \frac{\partial g}{\partial \bar z} = 0 = \frac{\partial h}{\partial z} \\ *\alpha = -i g\,dz + i h\,d\bar z & \Rightarrow \alpha + i*\alpha = 2g\,dz \end{array}\right)$$

By taking $\alpha = du$ with u given by the previous discussions:

prop i) $\forall P \in M$, $\exists$ meromorphic differential which is holomorphic on $M\setminus\{P\}$ and has singularity $\frac{1}{z^{n+1}}$ at P

ii) $\forall P, Q \in M$, $P \neq Q$, $\exists$ meromorphic differential which is holomorphic on $M\setminus\{P,Q\}$, has singularity $\begin{cases} \frac{-1}{z} & \text{at } P \\ \frac{1}{w} & \text{at } Q \end{cases}$

e.g. on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ $\quad \frac{1}{z} dz = w \, d(\frac{1}{w}) = -\frac{1}{w} dw$, $P = \infty$, $Q = 0$

For a meromorphic differential ω, we can also consider its order and residue.

$\omega = f(z) dz$, $z(p) = 0$

$$\mathrm{ord}_p \omega := \mathrm{ord}_0 f$$

$$\mathrm{Res}_p \omega = \mathrm{res}_0 f \quad : \text{the } z^{-1}\text{-coefficient in the Laurent expansion}$$

$$= \frac{1}{2\pi i} \int_\gamma \omega \quad \gamma: \text{any small circle around } p$$

[easy to check they are well-defined: independent of the choice of holomorphic coordinate]

<u>Prop</u> M: compact. ω: meromorphic differential

Then. $\sum_{p \in M} \mathrm{Res}_p \omega = 0$

pf:

For each singularity P_j, choose a small open disk $\mathcal{U}_j$ (ω is holomorphic here)

$$\sum_j \mathrm{Res}_{P_j} \omega = \frac{1}{2\pi i} \sum_j \int_{\partial \mathcal{U}_j} \omega = \underset{\text{orientation}}{-}\frac{1}{2\pi i} \int_{\partial(M \setminus \bigcup_j \mathcal{U}_j)} \omega = -\frac{1}{2\pi i} \iint_{M \setminus \bigcup_j \mathcal{U}_j} \underset{\to 0}{d\omega} = 0$$

#

existence

<u>thm</u> $P_1, \cdots, P_k$: distinct points on M. ← not required to be compact

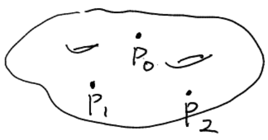
Given any $c_j \in \mathbb{C}$ with $\sum_{j=1}^k c_j = 0$. $\exists$ meromorphic differential ω on M with only singularities are P_j, each of order 1, and $\mathrm{res}_{P_j} \omega = c_j$

pf: Fix $P_0 \in M$, other than P_j's

$$\exists\, \omega_j = \begin{cases} (-\frac{1}{z} + \cdots) dz & \text{near } P_0 \\ (\frac{1}{w} + \cdots) dw & \text{near } P_j \end{cases} \Rightarrow \sum_{j=1}^k c_j \omega_j \text{ satisfies the desired property.}$$

<u>thm</u> any Riemann surface M admits a nontrivial meromorphic function

pf:



$$\exists\, \omega_1 = \begin{cases} (-\frac{1}{z} + \cdots) dz & \text{near } P_0 \\ (+\frac{1}{w} + \cdots) dw & \text{near } P_1 \end{cases}$$

$$\omega_2 = \begin{cases} (-\frac{1}{z} + \cdots) dz & \text{near } P_0 \\ (+\frac{1}{\zeta} + \cdots) d\zeta & \text{near } P_2 \end{cases}$$

$\Rightarrow$ ω_1 / ω_2 is a well-defined meromorphic function on M (P_1: pole, P_2: zero)

#

※ might have other poles and zeros.