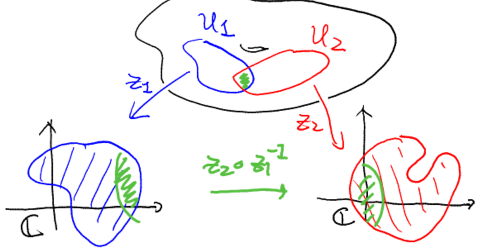
§1 definition, basic properties

Riemann surfaces : the theory of complex analysis on surfaces [FK, §I.1]

<u>defn</u> a Riemann surface is a (real) 2-dimensional manifold whose open set is modeled on open subsets of $\mathbb{C}$ and whose transition functions are biholomorphisms (conformal equivalences)

i) manifold : paracompact & Hausdorff, usually assume connectedness implicitly more interested in the compact ones

ii) Σ



$\forall p \in \Sigma$, $\exists\ \mathcal{U}_\alpha$ open neighborhood of p

$z_\alpha : \mathcal{U}_\alpha \longrightarrow z_\alpha(\mathcal{U}_\alpha) \subset \mathbb{C}$

homeomorphism to an open subset of $\mathbb{C}$

if $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$

$z_\beta \circ z_\alpha^{-1} : z_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to z_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$

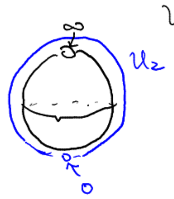
is a bijective analytic map. (between open subsets of $\mathbb{C}$)

iii) classically, a compact surface is said to be closed

<u>examples</u> i) the Riemann sphere, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \ni z$

$\mathcal{U}_1 = \{ z \in \hat{\mathbb{C}} \mid z \neq \infty \} \xrightarrow{z} \mathbb{C}$

$\mathcal{U}_2 = \{ z \in \hat{\mathbb{C}} \mid z \neq 0 \} \xrightarrow{w} \mathbb{C}$ $\qquad w = \begin{cases} \frac{1}{z} & z \in \mathbb{C} \setminus \{0\} \\ 0 & \infty \end{cases}$



$z(\mathcal{U}_1 \cap \mathcal{U}_2) = \mathbb{C} \setminus \{0\} \xrightarrow{w \circ z^{-1}} w(\mathcal{U}_1 \cap \mathcal{U}_2) = \mathbb{C} \setminus \{0\}$

$z \longmapsto \frac{1}{z}$: biholomorphism

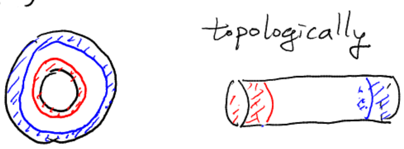
ii) recall $A(r_1, r_2) = \{ z \in \mathbb{C} \mid r_1 < |z| < r_2 \}$ annulus

$A(r_1, r_2) \cong A(\rho_1, \rho_2)$ if and only if $r_2/r_1 = \rho_2/\rho_1$

Fix $\varepsilon > 0$, a small scalar (say, $\varepsilon < \frac{1}{100}$)

Consider $A(1,2) \supset A(1, 1+\varepsilon)$

$\supset A(\frac{2}{1+\varepsilon}, 2)$



Choose two disjoint disks in $\hat{\mathbb{C}}$

$\mathcal{U}_1, \mathcal{U}_2 \cong D_1$

May assume both are biholomorphic to the unit disk (Riemann mapping)

Remove (the image of) the closure of $D_{1/1+\varepsilon}$

Glue $A(1,2)$ and the S^2 with two holes together

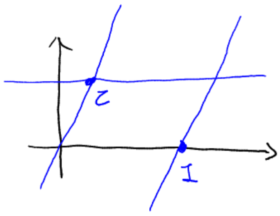
$\Rightarrow$ a Riemann surface with genus 1.

Similarly, one can construct a Riemann surface with genus g

rmk $A(1, 1+\varepsilon)$

$z \longmapsto \frac{1+\varepsilon}{z}$: biholomorphic & switches two boundary components.

iii) $\tau \in \mathbb{C}$, $\operatorname{Im} \tau > 0$ (simplest case, $\tau = i$)



$\mathbb{Z}^2 \curvearrowright \mathbb{C}$ by $(m,n) \cdot z = z + m + n\tau$

$\Rightarrow$ $\mathbb{C}/\mathbb{Z}^2$ is a Riemann surface

(topologically, it is a torus)

$\boxed{\text{HW}}$ construct a coordinate cover for it

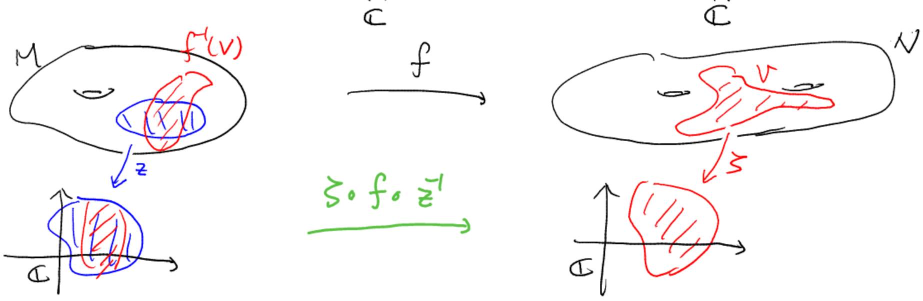
function / map M, N = two Riemann surfaces

$f : M \to N$. a continuous map is said to be holomorphic (or analytic)

if for any local coordinate $(\mathcal{U}, z)$ on M, and (V, ζ) on N

with $\mathcal{U} \cap f^{-1}(V) \neq \phi$, the map

$$\zeta \circ f \circ z^{-1} : \underset{\hat{\mathbb{C}}}{z(\mathcal{U} \cap f^{-1}(V))} \longrightarrow \underset{\hat{\mathbb{C}}}{\zeta(V)} \quad \text{is holomorphic}$$



Since the transition functions are biholomorphic, this notion is well-defined

Namely, given $p \in M$. $\rightsquigarrow$ $f(p) \in N$

if $\zeta \circ f \circ z^{-1}$ satisfies the Cauchy-Riemann equation at $z(p)$,

so does $\xi \circ f \circ w^{-1}$ at $w(p)$ for another coordinates

$(\tilde{\mathcal{U}} \ni p, w)$ on M and $(\tilde{V} \ni f(p), \xi)$ on N.

- holomorphic maps from M to $\mathbb{C}$ are called holomorphic functions
  holomorphic maps from M to $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ are called meromorphic functions.

thm M, N = Riemann surfaces, M: compact $f : M \to N$ holomorphic

Then, f is either constant or surjective

$\rightsquigarrow N$ is also compact

pf: if f is non-constant, $f(M)$ is open & connected

$\Rightarrow f(M) = N$ #

Cor M : compact Riemann surface, holomorphic function must be constant

$\mathcal{H}(M) = \mathbb{C}$: constants.

<u>degree</u> $f: M \to N$, non-constant holomorphic map between compact Riemann surfaces. $\forall P \in M$

$$\zeta \circ f \circ z^{-1} = \zeta(f(P)) + z^n h(z) \qquad (\text{assume } z(P)=0)$$

h: non zero, analytic on a neighborhood of 0

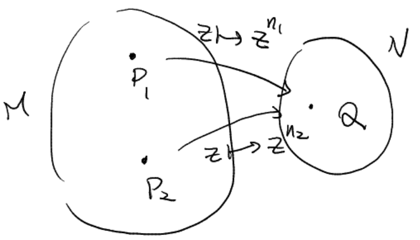
later on, we will abuse the notation and simply denote it by $f(z)$

n is independent of the choice of coordinate

is called the ramification number or multiplicity of f at P

$n-1$ is called the branch number, denote it by $b_f(P)$

prop $\exists\, m \in \mathbb{N}$. such that $\forall Q \in N$, it is assumed exactly m-times (counting multiplicities). Namely, $\sum_{P \in f^{-1}(Q)} (b_f(P) + 1) = m$



pf: For any $n \in \mathbb{N}$, consider $\Sigma_n \subset N = \{ Q \in N \mid \sum_{P \in f^{-1}(Q)} (b_f(P) + 1) \geq n \}$

1° Σ_n is open in N $\begin{cases} \text{by argument principle.} \\ \text{or. locally. } f(z) = z^n \text{ by inverse function theorem} \end{cases}$

2° Σ_n is closed in N

$Q_\alpha \in \Sigma_n$. $\lim_{\alpha \to \infty} Q_\alpha = Q \overset{?}{\in} \Sigma_n$

$b_f(P) > 0$ iff the derivative of $\zeta \circ f \circ z^{-1}$ is zero at $z(P)$

Since zeros of non-constant holomorphic function are discrete there are only finitely many ramification point (i.e. $P \in M$ w/ $b_f(P) > 0$)

Thus, there are only finitely many $Q \in N$ which are the image of some ramification point.

It follows that we may assume $Q_\alpha \neq$ image (ramification point)

$\Rightarrow$ each $f^{-1}(Q_\alpha)$ contains at least n distinct points.
$= \{ P_{\alpha 1}, P_{\alpha 2}, \cdots, P_{\alpha n} \}$

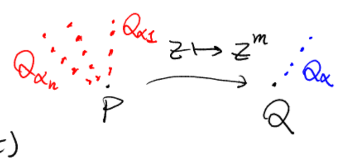
By passing to a subsequence (of α)
we may assume $P_{\alpha j} \xrightarrow{\alpha \to \infty} P_j$ for $j \in \{1, \cdots, n\}$

$\Rightarrow f(P_j) = \lim_{\alpha \to \infty} f(P_{\alpha j}) = \lim_{\alpha \to \infty} Q_\alpha = Q$

Since $P_{\alpha j} \neq P_{\alpha k}$ $\forall j \neq k$, $Q \in \Sigma_n$

[more: say. $P_1 = P_2 = \cdots = P_n = P$

i.e. $\exists$ a sequence of points $\to Q$ which has exactly (at least) n distinct pre-images.]



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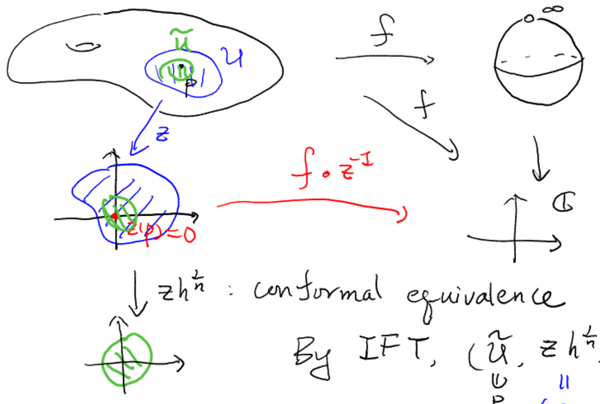
defn the number m in the proposition is called the degree of the map f, and f is called an m-sheeted (branch) cover

rmk f: non-constant meromorphic function on M $\left(M \xrightarrow{f} \hat{\mathbb{C}}\right)$

f actually determines the Riemann surface structure of M

pf: $p \in M$. assume $f(p) \neq \infty$ (otherwise look at $1/f$)



$f\circ z^{-1} = f(p) + z^n h(z)$

on a neighborhood of $z(p) = 0$

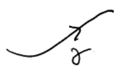
h: nowhere zero holomorphic function

$\leadsto$ so is $h^{\frac{1}{n}}$

By IFT, $(\tilde{\mathcal{U}}, zh^{\frac{1}{n}})$ is also a coordinate for M $\left(\tilde{\mathcal{U}} \ni p,\ zh^{\frac{1}{n}} = (f\circ z^{-1} - f(p))^{\frac{1}{n}}\right)$ ※

differentiation and integration on Riemann surface [FK, §I.3]

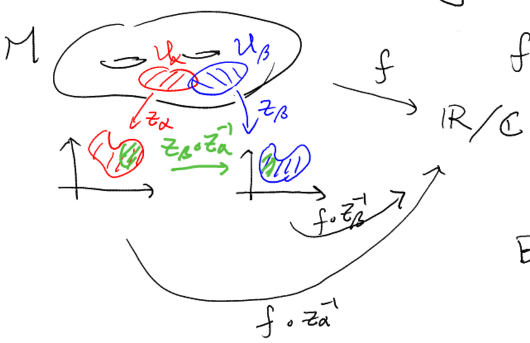
recall we study the line integral of $p\,dx + q\,dy$ or $f\,dz$



γ: continuous, piecewise smooth with direction/orientation $\leadsto$ $\int_\gamma p\,dx+q\,dy$ or $\int_\gamma f\,dz$

By the change of variable formula, the line integral is independent of the parametrization

What about the theory on Riemann surfaces?



$f: M \to \mathbb{R}$ or $\mathbb{C}$. notion of continuity $\leftsquigarrow$ M: top space

notion of holomorphic

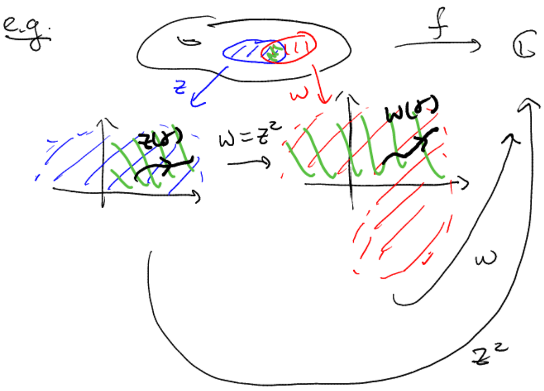
$\leftsquigarrow$ $z_\beta\circ z_\alpha^{-1}: z_\alpha(\mathcal{U}_\alpha\cap\mathcal{U}_\beta) \to z_\beta(\mathcal{U}_\alpha\cap\mathcal{U}_\beta)$ is a biholomorphism

By the same token, we have the notion of $C^1, C^2, \cdots C^k \cdots$ functions

Given $f: C^0$ (or C^k, C^∞) and $p\in M$ $\leadsto$ look at $f(p)$ (evaluation)

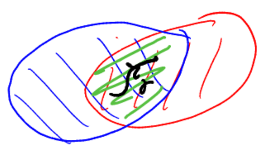
How about the line integral of f?

e.g.



say. $f\circ w^{-1} = w$

$f\circ z^{-1} = z^2$



easy to cook up examples so that

$\int_{w(\gamma)} w\,dw \neq \int_{z(\gamma)} z^2\,dz$

In fact since $w = z^2$, $\int_{w(\gamma)} w\,dw = \int_{z(\gamma)} z^2 (dz^2) = \int_{z(\gamma)} 2z^3\,dz$

(change of variable formula)

$\Rightarrow$ Those can be integrated over a 1-diml object (C^0, piecewise C^∞ curve) is a differential (a differential 1-form)

<u>defn</u> M : R-S. $\{(U_\alpha, z_\alpha)\}$: coordinate cover

a differential 1-form on M consists of a differential on each $z_\alpha(U_\alpha)$: $P_\alpha(z_\alpha)\,dx_\alpha + Q_\alpha(z_\alpha)\,dy_\alpha$ satisfying the change of variable formula

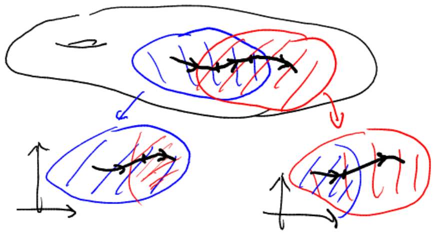
Namely, $P_\beta(z_\beta)\,dx_\beta + Q_\beta(z_\beta)\,dy_\beta \rightsquigarrow P_\beta(z_\beta(z_\alpha))\frac{\partial x_\beta}{\partial x_\alpha}dx_\alpha + P_\beta(z_\beta(z_\alpha))\frac{\partial x_\beta}{\partial y_\alpha}dy_\alpha$

$+ Q_\beta(z_\beta(z_\alpha))\frac{\partial y_\beta}{\partial x_\alpha}dx_\alpha + Q_\beta(z_\beta(z_\alpha))\frac{\partial y_\beta}{\partial y_\alpha}dy_\alpha$

$= P_\alpha(z_\alpha)\,dx_\alpha + Q_\alpha(z_\alpha)\,dy_\alpha$

(on $z_\alpha(U_\alpha \cap U_\beta)$)

It follows from the change of variable formula that the line integral of a differential 1-form over a directed path on M is well-defined.



Similarly we cannot consider the double integral of a function.

<u>defn</u> a 2-form is the following data:

$f_\alpha(z_\alpha)\,dx_\alpha \wedge dy_\alpha$ on each $z_\alpha(U_\alpha)$ such that

$$f_\beta(z_\beta)\,dx_\beta \wedge dy_\beta \rightsquigarrow f_\beta(z_\beta(z_\alpha))\underbrace{\frac{\partial(x_\beta, y_\beta)}{\partial(x_\alpha, y_\alpha)}}_{\text{Jacobian}}dx_\alpha \wedge dy_\alpha = f_\alpha(z_\alpha)\,dx_\alpha \wedge dy_\alpha$$

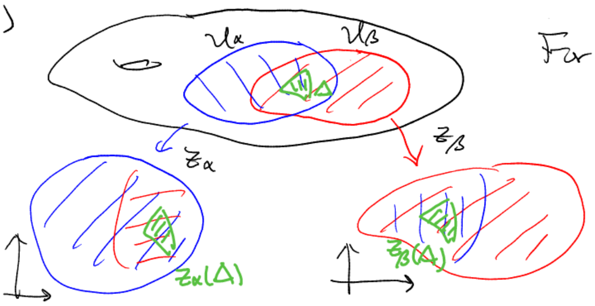
<u>rmk</u> i) $\wedge$: wedge product between two (locally defined) 1-forms

$dx_\alpha \wedge dx_\alpha = 0 = dy_\alpha \wedge dy_\alpha$. $dx_\alpha \wedge dy_\alpha = -dy_\alpha \wedge dx_\alpha$

$\boxed{\text{check}}$ well-defined on $U_\alpha \cap U_\beta$

$$\begin{array}{ccc} (U_\beta, z_\beta) & & (U_\alpha, z_\alpha) \\ dx_\beta,\ dy_\beta & \longrightarrow & \frac{\partial x_\beta}{\partial x_\alpha}dx_\alpha + \frac{\partial x_\beta}{\partial y_\alpha}dy_\alpha,\ \frac{\partial y_\beta}{\partial x_\alpha}dx_\alpha + \frac{\partial y_\beta}{\partial y_\alpha}dy_\alpha \\ \downarrow & \circlearrowleft & \downarrow \\ dx_\beta \wedge dy_\beta & \longrightarrow & \frac{\partial(x_\beta, y_\beta)}{\partial(x_\alpha, y_\alpha)}dx_\alpha \wedge dy_\alpha \end{array}$$

ii)



For a 2-form $\{f_\alpha(z_\alpha)\,dx_\alpha \wedge dy_\alpha\}_\alpha$ on M

$$\iint_{z_\alpha(\Delta)} f_\alpha(z_\alpha)\,dx_\alpha \wedge dy_\alpha := \iint_{z_\alpha(\Delta)} f_\alpha(z_\alpha)\,dx_\alpha dy_\alpha$$

↑ Riemann / Lebesgue integral

$\overset{\boxed{\text{check}}}{=} \iint_{z_\beta(\Delta)} f_\beta(z_\beta)\,dx_\beta \wedge dy_\beta$ for any $\Delta \subset U_\alpha \cap U_\beta$.

	in terms of local coordinate	
functions	$f(x,y)$	evaluate at points
differential (1-forms)	$p(x,y)\,dx + q(x,y)\,dy$	line integral
2-forms	$f(x,y)\,dx\wedge dy$	double integral over domain

they are called "chains" (will explain later)

Stokes' theorem (general form of the fundamental theorem of calculus, Gauss theorem, Green theorem)

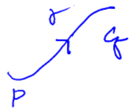
$$\int_{\partial D} \omega = \int_D d\omega \quad \leftarrow \text{certain derivative of } \omega$$

∂D: boundary of D (oriented counterclockwisely)

defn $d(p\,dx + q\,dy) = \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\wedge dy$

$$\int_{\partial\gamma} f = \int_\gamma df$$

$\int_{\partial\gamma} f = f(q) - f(p)$



defn $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

claim d is well-defined

df on $U\cap V$ $\quad (U, z=x+iy)$, $(V, w=u+iv)$

$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv$

$= \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)$

$=$ (chain rule) $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$\boxed{\text{check}}$ d on 1-form is well-defined

prop $d^2 f = 0$

pf: $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$ ... ※

Riemann surface: Cauchy-Riemann equation

$(U, z=x+iy)$ as complex analysis, introduce

$$\begin{cases} \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{cases} \qquad \begin{cases} dz = dx + i\,dy \\ d\bar z = dx - i\,dy \end{cases}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar z}d\bar z$$

$$d(p\,dx + q\,dy) = d\left(\tfrac{1}{2}(p - qi)\,dz + \tfrac{1}{2}(p+qi)\,d\bar z\right)$$
$$= \frac{\partial}{\partial \bar z}\left(\frac{p-qi}{2}\right)d\bar z\wedge dz + \frac{\partial}{\partial z}\left(\frac{p+qi}{2}\right)dz\wedge d\bar z = \frac{1}{2}\left(\frac{\partial}{\partial z}(p+qi) - \frac{\partial}{\partial \bar z}(p-qi)\right)dz\wedge d\bar z$$

We can decompose d as follows: $d = \partial + \bar\partial$

on function f: $\partial f = \frac{\partial f}{\partial z}dz \qquad \bar\partial f = \frac{\partial f}{\partial \bar z}d\bar z$

on 1-form $\alpha\,dz + \beta\,d\bar z$,

$dz\wedge dz = (dx + i\,dy)\wedge(dx+i\,dy) = 0$

$\partial(\alpha\,dz + \beta\,d\bar z) := (\partial\alpha)\wedge dz + (\partial\beta)\wedge d\bar z = \frac{\partial\beta}{\partial z}dz\wedge d\bar z$

$\bar\partial(\alpha\,dz + \beta\,d\bar z) := (\bar\partial\alpha)\wedge dz + (\bar\partial\beta)\wedge d\bar z = -\frac{\partial\alpha}{\partial \bar z}dz\wedge d\bar z$

Well-defined? $\quad (U, z=x+iy) \quad (V, w=u+iv)$

$u(x,y)$, $v(x,y)$ satisfy the Cauchy Riemann equation

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases} \iff \frac{\partial w}{\partial \bar z} = 0$$

A direct computation shows that

$$dw = \frac{\partial w}{\partial z} dz \qquad \frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\frac{\partial}{\partial w}$$

$$d\bar{w} = \overline{\left(\frac{\partial w}{\partial z}\right)} d\bar{z} \qquad \frac{\partial}{\partial \bar{z}} = \overline{\left(\frac{\partial w}{\partial z}\right)}\frac{\partial}{\partial \bar{w}}$$

no $d\bar{z}$ no $\frac{\partial}{\partial \bar{w}}$

relies on the Cauchy-Riemann equation

More explicitly, $\bar{\partial} f = \boxed{\frac{\partial f}{\partial \bar{z}} d\bar{z}} = \frac{\partial f}{\partial \bar{w}} \overline{\left(\frac{\partial w}{\partial z}\right)} d\bar{z}$

$$= \boxed{\frac{\partial f}{\partial \bar{w}} d\bar{w}}$$

<u>more about 1-forms</u> on Riemann surfaces

ideal / motivation $\quad f$: holomorphic on $\mathcal{U} \subset \mathbb{C} \Rightarrow f = u + iv$

conjugate harmonic

- $df = \partial f = f'(z)\, dz$

  still holomorphic

- $du = u_x\, dx + u_y\, dy \qquad dv = v_x\, dx + v_y\, dy$

  $\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -u_y dx + u_x dy$

  line integral recover u line integral recover v

- harmonic function : f : function $\quad d^2 f = 0 \qquad d = \partial + \bar{\partial}$

  $$d^2 = (\partial + \bar{\partial})^2 = \cancel{\partial^2}^{\,0} + \partial\bar{\partial} + \bar{\partial}\partial + \cancel{\bar{\partial}^2}^{\,0}$$

  $$\bar{\partial}\partial f = \bar{\partial}\left(\frac{\partial f}{\partial z}\right) dz$$

  $$= \frac{\partial^2 f}{\partial \bar{z} \partial z} d\bar{z} \wedge dz = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)(+2i\, dx \wedge dy) = \frac{i}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy$$

  <u>defn</u> f is harmonic if $\bar{\partial}\partial f = 0$

- harmonic 1-form : $\quad f$: harmonic function

  $$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \qquad df = 0$$

  still harmonic

  examine the differential for constructing the conjugate harmonic

  $$* df = -\frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial x} dy$$

  $$d * df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy = 0$$

<u>defn</u> i) $*(p\, dx + q\, dy) := -q\, dx + p\, dy$

$\qquad *(\alpha\, dz + \beta\, d\bar{z}) := -i\alpha\, dz + i\beta\, d\bar{z}$

ii) a 1-form ω is said to be harmonic if ω is closed & coclosed, i.e. $d\omega = 0$ & $d * \omega = 0$

<u>prop</u> ω is harmonic if and only if ω is locally the d(harmonic)

$\forall p \in M$ $\exists$ nbd and f: defined on it such that $df = \omega$ (on the nbd)

pf: $d\omega = 0 \Rightarrow \omega = df$ locally

by line integral

$d * \omega = 0 \Rightarrow f$: harmonic ※

$= d * df = 0$

<u>prop</u> $d * d = -2i\bar{\partial}\partial = 2i\partial\bar{\partial}$ (on functions)

<u>rmk</u> we will see later that harmonic 1-forms are related to the topology of a Riemann surface.

- holomorphic differential $\quad$ <u>defn</u> a 1-form is called a holomorphic differential if on each coordinate chart it is $\alpha(z)\, dz$ for $\alpha(z)$: holomorphic

1° $\underline{\alpha(w)\,dw} = \underline{\alpha(w(z))\frac{\partial w}{\partial z}}\,dz$

same holomorphicity

2° $d(\alpha(z)\,dz) = \cancel{\frac{\partial \alpha}{\partial \bar z}}\,d\bar z\wedge dz = 0 \Rightarrow \alpha(z)\,dz = df(z)$ locally

or equivalently, $\alpha(z) = \frac{\partial f}{\partial z}$ locally

Since $\frac{\partial f}{\partial \bar z}$ is holomorphic, so is f

prop a 1-form is holomorphic if it is locally d(holomorphic function)

3° $f = u + iv$ $\quad df = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)dy$

f: holomorphic $\quad = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)dx + \underline{\left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial x}\right)}dy$

$= \frac{1}{2}\frac{\partial u}{\partial z}\,dz \qquad$ $\overset{"}{=} i\frac{\partial u}{\partial z}$

prop u: harmonic $\Rightarrow \partial u$ is a holomorphic differential

4° ω: holomorphic 1-form $\overset{\text{locally}}{=} d(u+iv)$ $\rightarrow$ harmonic 1-form

$= du + i * du$

prop It is true globally, Namely, ω is a holomorphic differential if and only if $\omega = \alpha + i * \alpha$ for some harmonic 1-form

$(f = u + iv)$

pf: $\Rightarrow$) ω: holomorphic $\Rightarrow$ ω & $\bar\omega$ are harmonic

$\omega \xrightarrow{*} -i\omega$, $\quad \bar\omega \xrightarrow{*} i\bar\omega$

take $\alpha = \frac{\omega - \bar\omega}{2} \Rightarrow$ DONE.

$\Leftarrow$) α: harmonic $\overset{\text{locally}}{=} du$ $\quad u$: harmonic

locally $\exists$ conjugate harmonic v

$\Rightarrow \alpha + i*\alpha = du + i\,dv = d(u+iv)$: holomorphic ※

5° $*$: an internal structure on 1-forms, induced by the Riemann surface structure.

More importantly, $p\,dx + q\,dy \xrightarrow{*} -q\,dx + p\,dy$

$(p\,dx + q\,dy)\wedge *\overline{(p\,dx+q\,dy)} = (p\,dx+q\,dy)\wedge(-\bar q\,dx + \bar p\,dy)$

$= (|p|^2 + |q|^2)\,dx\wedge dy$

in z-coordinate $\quad \alpha\,dz + \beta\,d\bar z \xrightarrow{\text{conjugate}} \bar\beta\,dz + \bar\alpha\,d\bar z \xrightarrow{*} -i\bar\beta\,dz + i\bar\alpha\,d\bar z$

wedge

$i(|\alpha|^2 + |\beta|^2)\,dz\wedge d\bar z = 2(|\alpha|^2+|\beta|^2)\,dx\wedge dy$