(1) (a) $d\eta = 0$, $d*\eta = 0$

(b) $\eta \rightsquigarrow \frac{1}{2}(\eta + i*\eta)$

$\eta = f\,dz + g\,d\bar{z}$ $\quad d\eta = 0 \Rightarrow \frac{\partial}{\partial \bar{z}} f - \frac{\partial}{\partial z} g = 0$

$*\eta = -i f\,dz + i g\,d\bar{z}$ $\quad d*\eta = 0 \Rightarrow \frac{\partial}{\partial \bar{z}} f + \frac{\partial}{\partial z} g = 0$ $\quad \Rightarrow \begin{cases} \frac{\partial f}{\partial \bar{z}} = 0 \\ \frac{\partial g}{\partial z} = 0 \end{cases}$

η : harmonic $\Rightarrow$ f: holomorphic, g: anti-holomorphic

$\Rightarrow \frac{1}{2}(\eta + i*\eta) = f\,dz$: holomorphic $\rightsquigarrow$ No $d\bar{z}$-component and annihilated by $\bar{\partial}$

(2) By R-R, $r(P^{-1}) - i(P) = \deg(P) + 1 - g = 2 - g$

$\Rightarrow i(P) = g - 2 + r(P^{-1})$

$L(P^{-1}) = \{ f : \text{meromorphic} \mid (f) \geq P^{-1} \}$

if $f \neq$ constant function, f has a simple pole at P

$\Rightarrow f : M \to \hat{\mathbb{C}}$, degree 1 holomorphic map

hence a biholomorphic $\rightarrow\!\leftarrow$ with $g \geq 1$

$\Rightarrow L(P^{-1}) = \{\text{constant functions}\} \Rightarrow r(P^{-1}) = 1 \Rightarrow i(P) = g - 1$

(3)(a) $g = 0 \Rightarrow \deg(\omega) = -2$ by direct counting

$g \geq 1 \Rightarrow$ Choose a nonzero holomorphic differential, and apply R-R

$r((\omega)^{-1}) = \deg(\omega) + 1 - g + i((\omega))$

By Serre duality, $\begin{cases} r((\omega)^{-1}) = i(1) = g \\ i((\omega)) = r(1) = 1 \end{cases}$

$\Rightarrow g = \deg(\omega) + 1 - g + 1 \Rightarrow \deg(\omega) = 2g - 2$

(b) Choose a non-zero holomorphic differential ω

$\eta = f(z)(dz)^3 \Leftrightarrow h = \frac{\eta}{\omega^3}$ is a meromorphic differential

η: holomorphic $\Leftrightarrow (\eta) \geq 1 \Leftrightarrow (h)(\omega)^3 \geq 1 \Leftrightarrow (h) \geq (\omega)^{-3}$

Hence, $\{\text{holomorphic cubic differential}\} \Leftrightarrow L((\omega)^{-3})$

By R-R, $r((\omega)^{-3}) = \deg(\omega)^3 + 1 - g + i((\omega)^3)$

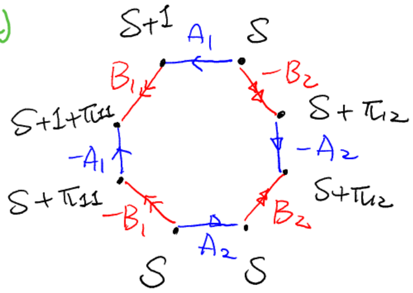
$\deg(\omega)^3 = 3\deg(\omega) = 3(2g-2) = 6g - 6$

By Serre duality, $i((\omega)^3) = r((\omega)^2) = 0 \leftarrow$ vanishing

$\deg(\omega)^2 = 2(2g-2) > 0$

$\Rightarrow r((\omega)^{-3}) = 6g - 6 + 1 - g = 5(g-1)$

(4) (a)

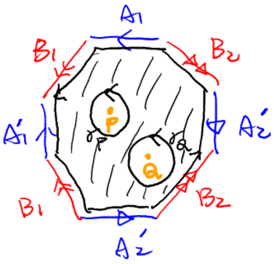


Since $\int_{A_1} \zeta_1 = 1$, $\int_{A_2} \zeta_1 = 0$,

$\int_{B_1} \zeta_1 = \pi_{11}$, $\int_{B_2} \zeta_1 = \pi_{12}$

we have

(b)



Since f & z are holomorphic in the shaded region.

$$\overset{\text{Stokes}}{\Longrightarrow} \quad \int_{\gamma_P} fz + \int_{\gamma_Q} fz = \int_{\partial(\text{polygon})} fz$$

$$LHS = 2\pi i \left(f(P)\, \mathrm{Res}_P z + f(Q)\, \mathrm{Res}_Q (z) \right)$$

$$= 2\pi i \left(f(P) - f(Q) \right) = 2\pi i \int_Q^P \zeta_1$$

$$RHS = \left(\int_{A_1} fz - \int_{A_1'} fz \right) + (\text{similar})$$

A_1 & A_1' : same curve on M. $z|_{A_1} = z|_{A_1'}$

By part (a) $f|_{A_1} - f|_{A_1'} = -\pi_{11}$

$$\rightsquigarrow = -\pi_{11} \cancel{\int_{A_1} z}^{\,0} + \int_{B_1} z - \pi_{12} \cancel{\int_{A_2} z}^{\,0} = \int_{B_1} z$$

(5) (a) Since there is only one pole of order 2

$\Rightarrow f : M \to \hat{\mathbb{C}}$. has degree 2

Hence. $\# \{ f^{-1}(c) \}$ is either 1 or 2

(b) By R-H.

$$\chi(M) = \deg(f)\, \chi(\hat{\mathbb{C}}) - \sum_{Q \in M} b_f(Q)$$

$$0 = 2 \cdot 2 - \sum_{Q \in M} b_f(Q)$$

$$\Rightarrow \sum_{Q \in M} b_f(Q) = 4$$

Since $b_f(Q) \le \deg(f) - 1$, $b_f(Q)$ can only be 1 (or 0)

$\Rightarrow \exists$ 4 points on M. with $b_f(Q) = 1$

(one of them is P)

Say Q_1, Q_2, Q_3, P

Since Q_j already contributes 2 to the (multiplicity) counting of $f^{-1}(f(Q_j))$, there is no other point in M being mapped to $f(Q_j)$ $\Rightarrow$ 3 - values in $\mathbb{C}$.

which are $f(Q_1)$, $f(Q_2)$, $f(Q_3)$