RIEMANN SURFACE
HOMEWORK 9

DUE: TUESDAY, MAY 17

(1) If $f(z)$ is holomorphic, prove that $|f(z)|^\alpha$ ($\alpha \geq 0$) and $\log(1 + |f(z)|^2)$ are subharmonic.

(2) Let $f : M \to N$ be a (non-constant) holomorphic map. If the Green's function exists for N, show that the Green's function exists for M and $g_M(p, q) \leq g_N(f(p), f(q))$.

(3) Let $\Omega \subset M$ be an open set. For a point $q \in \partial\Omega$, choose a coordinate at q with $z(q) = 0$. A *subharmonic barrier* at q is a subharmonic function β defined by $\Omega' = \Omega \cap \{|z| < \rho\}$ for some $\rho > 0$ such that

- $\beta(p) < 0$ for any $p \in \Omega'$, and $\beta(p) \to 0$ as $p \to q$;
- $\limsup_{p \to q'} \beta < 0$ for any $q' \in \partial\Omega \cap \{0 < |z| < \rho\}$.

(a) Prove that β can be used to construct a subharmonic function on Ω with the above properties. (Hint: Let $-2C = \sup_{\Omega \cap \{\rho/2 \leq |z| \leq 3\rho/4\}} \beta$. Consider

$$\tilde{\beta} = \begin{cases} \max(\beta, -C) & q' \in \Omega \cap \{0 < |z| < \rho/2\} \\ -C & \text{otherwise} \end{cases} .)$$

(b) For the unit disk in $\mathbb{C}$, construct a subharmonic barrier at each point in the unit disk.

(c) For the completement of the disk in $\mathbb{C}$, construct a subharmonic barrier at each point in the unit disk.