RIEMANN SURFACE
HOMEWORK 7

DUE: TUESDAY, APRIL 19

(1) Suppose that $g \geq 1$. Show that $r(P^{-1}) = 1$.

As a consequence, the first gap n_1 (at any P) is 1.

(2) Given any $P \in M$, a positive integer n is called a non-gap if $r(P^{-n}) = r(P^{-(n-1)}) + 1$. Suppose that i, j are both non-gaps. Show that $i + j$ is also a non-gap.

In other words, non-gaps constitute a semi-group under addition.

(3) Suppose that $g \geq 2$. Denote the first g non-gaps by $\alpha_1 < \alpha_2 < \cdots < \alpha_g$. If $\alpha_1 > 2$, prove that there exists some $k \in \{1, 2, \ldots, g-1\}$ such that

$$\alpha_k + \alpha_{g-k} > 2g \ .$$

<u>Hint</u>: It is straightforward when $g < 4$. Thus, consider the case when $g \geq 4$, and assume

$$\alpha_k + \alpha_{g-k} = 2g \qquad (\bowtie)$$

for any k. Since $\alpha_1 > 2$ and $g \geq 4$, there must exist some non-gap not divisible by α_1. Let the smallest one be α_{r+1}. That is to say, the first $(r+1)$ non-gaps are

$$\alpha_1 \ , \alpha_2 = 2\alpha_1 \ , \quad \cdots \quad , \alpha_r = r\,\alpha_1 \ , \alpha_{r+1} \ .$$

By $(\bowtie)$,

$$\alpha_{g-1} = 2g - \alpha_1 \ , \quad \cdots \quad , \alpha_{g-r} = 2g - r\,\alpha_1 \ , \alpha_{g-r-1} = 2g - \alpha_{r+1} \ .$$

What can you say about the non-gap $\alpha_1 + \alpha_{g-r-1}$?

(4) A (*holomorphic*) *quadratic differential* consists of the following data: a holomorphic function $f_\alpha(z_\alpha)$ on each coordinate chart (U_α, z_α) such that

$$f_\beta(z_\beta \circ z_\alpha^{-1}) \left(\frac{\mathrm{d}z_\beta}{\mathrm{d}z_\alpha} \right)^2 = f_\alpha(z_\alpha) \quad \text{when } U_\alpha \cap U_\beta \neq \varnothing \ .$$

It is convenient to write a quadratic differential as

$$f_\alpha(z_\alpha)\,(\mathrm{d}z_\alpha)^2 \ .$$

Suppose that $g > 1$. Calculate the dimension of the space of holomorphic quadratic differentials.

<u>Hint</u>: Fix a nonzero holomorphic differential ω. Recall that $\deg(\omega) = 2g - 2$. Let μ be a holomorphic quadratic differential. Note that $\frac{\mu}{\omega^2}$ is a meromorphic function whose divisor is greater than or equal to $(\omega)^{-2}$. On the other hand, $f\omega^2$ defines a holomorphic quadratic differential for any $f \in L((\omega)^{-2})$.