RIEMANN SURFACE
HOMEWORK 6

DUE: TUESDAY, APRIL 12

(1) Prove that the degree of a canonical divisor is $2g-2$.

<u>Hint</u>: It is straightforward when $g=0$. Thus, it suffices to consider the case when $g \geq 1$. We know that there exist non-trivial *holomorphic differentials*. Take one of them, and denote it by ζ. Since ζ is holomorphic, $(\zeta) \geq 1$. What does the Riemann–Roch theorem say for the divisor (ζ)?

(2) Given any finite (distinct) points, $P_1, \cdots, P_n$ on M. Prove that there exists a meromorphic differential ω which is holomorphic and nonzero at these points.

<u>Hint</u>: We use the Riemann–Roch theorem (for any divisor):

$$r(\mathfrak{A}^{-1}) = \deg \mathfrak{A} - g + 1 + i(\mathfrak{A}) \ .$$

For each $j \in \{1, \ldots, n\}$, we would like to construct a meromorphic differential ω_j which is holomorphic at $P_1, \cdots, P_n$, and is nonzero at P_j and zero at other P's. Then $\omega = \sum_{j=1}^{n} \omega_j$ would admit the required property. To find such ω_j, pick a point Q other than P's, and consider the Riemann–Roch theorem for $\mathfrak{A} = Q^{-(n+1)} P_1 \cdots P_n$ and $\mathfrak{A} P_j^{-1}$.

(3) Use (1) and (2) to give another proof of the Riemann–Hurwitz formula.

<u>Hint</u>: Suppose that $f : M_1 \to M_2$ is a non-constant holomorphic map. Choose a non-trivial meromorphic differential ω on M_2. The *pull-back* of ω by f is a meromorphic differential on M_1, and is denoted by $f^*\omega$. It is constructed as follows: Write the map f in terms of local coordinate:

$$z \mapsto w = f(z)$$

where f is a locally defined holomorphic function. It is common to abuse the notation here. Then, express ω in terms of the w-coordinate,

$$\omega = g(w)\, \mathrm{d}w$$

where g is a locally defined meromorphic function. Then, the pull-back of ω by f is

$$g(f(z))\, f'(z)\, \mathrm{d}z$$

in terms of the z-coordinate. It is not hard to check the well-definedness (namely, the transition rule). Thus, $f^*\omega$ is a meromorphic differential on M_1. Choose ω so that it is *holomorphic and nonzero at the images of the branch points of* f. Then analyze the relations between the degree of ω and that of $f^*\omega$.