RIEMANN SURFACE
HOMEWORK 5

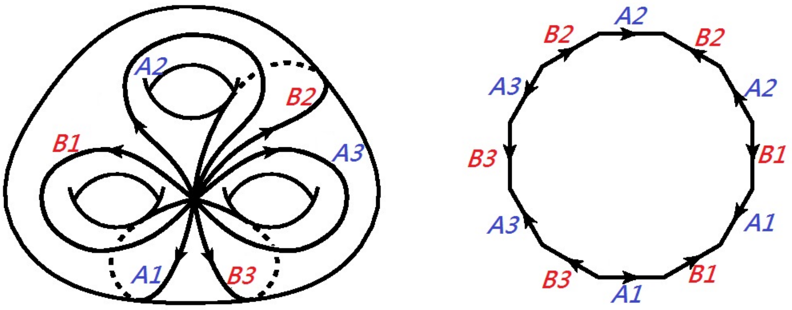
DUE: THURSDAY, APRIL 7

(1) Let M be a compact Riemann surface with genus $g \geq 1$. Fix a $2g$-loops-cut: $A_1, \cdots, A_g$, $B_1, \cdots, B_g$. Let $\zeta_1, \cdots, \zeta_g$ be the basis of the space of holomorphic differentials satisfying the normalizations

$$\int_{A_j} \zeta_k = \delta_{kj} \ .$$

Denote their B-periods by

$$\int_{B_j} \zeta_k = \pi_{kj} \ .$$



(a) Let τ be a meromorphic differential on M such that the singularities of τ do not lie on the loops-cuts. Show that there exists a holomorphic differential η such that

$$\int_{A_j} (\tau - \eta) \in i\mathbb{R} \qquad \text{and} \qquad \int_{B_j} (\tau - \eta) \in i\mathbb{R} \qquad (\dagger)$$

for $j \in \{1, \ldots, g\}$.

(b) Let P and Q be two distinct points in $M \backslash \{\text{loops-cuts}\}$. Suppose that ω is a meromorphic differential on M such that

$$\omega \text{ is holomorphic on } M \backslash \{P, Q\} \ ,$$
$$\operatorname{ord}_P \omega = -1 = \operatorname{ord}_Q \omega \ ,$$
$$\operatorname{Res}_P \omega = 1 \ , \quad \operatorname{Res}_Q \omega = -1 \ ,$$
$$\int_{A_j} \omega \in i\mathbb{R} \ , \quad \int_{B_j} \omega \in i\mathbb{R}$$

for $j \in \{1, \ldots, g\}$. Prove that

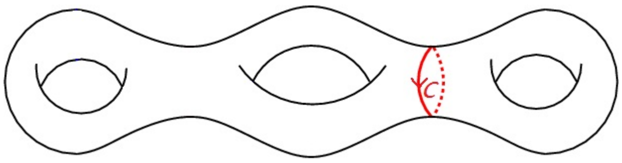
$$2\pi i \int_Q^P \zeta_j = \int_{B_j} \omega - \sum_{k=1}^{g} \left(\pi_{jk} \int_{A_k} \omega \right)$$

for $j \in \{1, \ldots, g\}$. Here, $\int_Q^P$ is taken over any path from Q to P disjoint from the loops-cuts.

(2) Given a directed, smooth curve C on a Riemann surface M. We discuss how to construct a closed differential η_C such that

$$\int_M \omega \wedge \eta_C = -\int_C \omega \quad \text{for any closed differential } \omega \ .$$

What can you say about η_C for the curve C in the following picture?



(3) Let D be the unit disk in $\mathbb{C}$. Given two non-constant holomorphic functions on D, f and g, we define

$$W(f,g) = \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} \ .$$

Clearly, $W(f,g)$ is also a holomorphic function on D. Prove that f and g are linearly independent if and only if $W(f,g)$ is not identically zero.