RIEMANN SURFACE
HOMEWORK 4

DUE: TUESDAY, MARCH 29

(1) Let M_1 and M_2 be two compact Riemann surfaces, and $f : M_1 \to M_2$ be a non-trivial holomorphic map. Denote the genus of M_1 and M_2 by g_1 and g_2, respectively. Define the *total branching number* of f by

$$B(f) = \sum_{P \in M_1} b_f(P) \ .$$

Since f is non-trivial and M_1 is compact, the sum on the right hand side is a finite sum.

(a) Show that B is always an even integer.

(b) Analyze all the possible situations of f and M_2 for $g_1 = 0$.

(c) Suppose that there is no branch point, i.e. $B = 0$. Prove that $g_1 \geq g_2$.

Hint: Use the Riemann–Hurwitz formula.

(2) Let M be a compact Riemann surface. A (non-trivial) meromorphic function f is defined to be a holomorphic map to $\hat{\mathbb{C}}$. Due to the discussion for $\deg(f)$, we know that

$$\deg(f) = \sum_{P \in f^{-1}(0)} (b_f(P) + 1) = \sum_{P \in f^{-1}(\infty)} (b_f(P) + 1) \ .$$

In other words, the total number of zeros of f is the same as that of poles (counting multiplicity).

Let us move on to the discussion of meromorphic differentials. Recall that the *order* of a meromorphic differential at some point p is defined to be $n \in \mathbb{Z}$ if

$$\omega = z^n \, h(z) \, \mathrm{d}z \quad \text{ in terms of a coordinate with } z(P) = 0$$

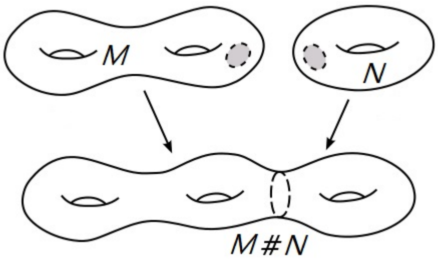
where $h(z)$ is a locally defined, nowhere zero holomorphic function. We denote it by $\mathrm{ord}_P \, \omega = n$.

Suppose that ω is a (non-trivial) meromorphic differential on the Riemann sphere $\hat{\mathbb{C}}$. Prove that

$$\sum_{P \in \hat{\mathbb{C}}} \mathrm{ord}_P \, \omega = -2 \ .$$

Hint: You may start with the case when $\omega_0 = z^{-1}\mathrm{d}z = -w^{-1}\mathrm{d}w$. Then use the fact that ω/ω_0 is a meromorphic function on $\hat{\mathbb{C}}$.

(3) Given two compact, oriented surfaces, the (*oriented*) *connected sum* is the procedure to construct another surface as shown in the following picture.



To explain more, choose two open sets in M and N such that they are diffeomorphic to the unit disk in $\mathbb{R}^2$. Denote the open sets by $U \subset M$ and $V \subset N$. Let $U' \subset U$ and $V' \subset V$ be the image of the disk of radius 1/2 under (the inverse of) the coordinate map. Then, the connected sum is

$$M\#N = \frac{(M\backslash\overline{U'})\coprod(N\backslash\overline{V'})}{\sim}$$

where we identify the points in $U\backslash\overline{U'}$ with the points in $V\backslash\overline{V'}$ by

$$\begin{array}{ccc} U\backslash\overline{U'} \cong D_1\backslash\bar{D}_{\frac{1}{2}} & \longrightarrow & D_1\backslash\bar{D}_{\frac{1}{2}} \cong V\backslash\overline{V'} \\ re^{i\theta} & \longmapsto & \dfrac{1}{2r}e^{-i\theta} \end{array}.$$

(a) Find out the relation between $\chi(M)$, $\chi(N)$ and $\chi(M\#N)$.

(b) Find out the relation between $\mathrm{H}_1(M)$, $\mathrm{H}_1(N)$ and $\mathrm{H}_1(M\#N)$, as well as the intersection theory.