RIEMANN SURFACE
HOMEWORK 3

DUE: TUESDAY, MARCH 22

(1) Let $U \subset \mathbb{C}$ be an open and connected region, and $f : U \to C$ be a nowhere zero holomorphic function. Show that $\log|f|$ is a harmonic function.

(2) This exercise is related to [FK,§II.4.1]. Let M be a compact Riemann surface, and P be a point on M. Choose a local coordinate z on a neighborhood of P such that $z(P) = 0$ and its image contains $\overline{D_a} = \{z \in \mathbb{C} \mid |z| \leq a\}$. We may regard D_a as an open neighborhood of P in M.

Let

$$\mathcal{F} = \left\{ v : \text{smooth function on } M\backslash\{P\} \;\middle|\; v|_{D_a\backslash\{0\}} = \frac{1}{z^n} + \frac{\bar{z}^n}{a^{2n}} + \tilde{v}(z) \ , \right.$$
$$\left. \text{where } \ \tilde{v} \ \text{ is a smooth function on } \ D_a \right\} .$$

For any $v \in \mathcal{F}$, the following integral is always finite:

$$\mathrm{D}[v] = \left|\mathrm{d}(v - \frac{1}{z^n} - \frac{\bar{z}^n}{a^{2n}})\right|^2_{D_a} + |\mathrm{d}v|^2_{M\backslash D_a} \ .$$

Prove that the function u given by [FK, Theorem II.4.1] minimizes D, namely,

$$\mathrm{D}[u] \leq \mathrm{D}[v] \quad \text{for any} \quad v \in \mathcal{F} \ .$$

(3) According to [FK, Theorem II.5.1], any Riemann surface admits a meromorphic differential with only one pole (at some $P \in M$), and the singularity is $\frac{1}{z^2}$ (where z is a coordinate on a neighborhood of P so that $z(P) = 0$).

Now suppose that M is a Riemann surface, and is *diffeomorphic* to a sphere. Fix $Q \in M\backslash\{P\}$. Denote the above meromorphic differential by ω.

(a) Prove that there exists a holomorphic function f on $M\backslash\{P\}$ such that $\mathrm{d}f = \omega$ (on the non-compact Riemann surface $M\backslash\{P\}$).
Hint: The function is in a way the "integral" of ω. Since M is diffeomorphic to a sphere, $M\backslash\{P\}$ is diffeomorphic to $\mathbb{R}^2$. the Jordan curve theorem asserts that any closed curve[1] in $\mathbb{R}^2$ is the boundary of some bounded region in $\mathbb{R}^2$.

(b) Study the behavior of f on a neighborhood of P.

(c) Regard f as a holomorphic map from M to $\hat{\mathbb{C}}$. What is the degree of f?

[1]For simplicity, we consider only continuous, piecewise smooth curves.

As a consequence, you find that f in fact gives a biholomorphism between M and $\hat{\mathbb{C}}$. In other words, a Riemann surface which is diffeomorphic to a sphere must be the Riemann sphere. This is not true for higher genus surfaces. For instance, there are Riemann surface structures on a torus which are not biholomorphic. (3) of Homework 1 is in a way the baby version of it.