RIEMANN SURFACE
HOMEWORK 2

DUE: TUESDAY, MARCH 15

(1) If f is a holomorphic function defined on some domain containing $\{z \in \mathbb{C} \mid |z| \leq 1\}$ and u is a smooth function whose support is contained in the unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$, then

$$\int\!\!\int_D f\left(\frac{\partial}{\partial \bar{z}} u\right) \mathrm{d}x\,\mathrm{d}y = -\int\!\!\int_D \left(\frac{\partial}{\partial \bar{z}} f\right) u\,\mathrm{d}x\,\mathrm{d}y = 0 \ .$$

Prove the following version of the Weyl's lemma. Let φ be a measurable, square-integrable function on D such that

$$\int\!\!\int_D \varphi\left(\frac{\partial}{\partial \bar{z}} u\right) \mathrm{d}x\,\mathrm{d}y = 0$$

for any smooth function u with $\operatorname{supp} u \subset D$. Prove that φ is a holomorphic function, i.e. φ is equal almost everywhere to a holomorphic function on D.

Here are some hints.

- Consider

$$\eta(w) = \frac{-1}{\pi}\int\!\!\int_{\mathbb{C}} \frac{u(z)}{z - w}\,\mathrm{d}x\,\mathrm{d}y \ .$$

  Is η a smooth function? What is $\frac{\partial}{\partial \bar{w}}\eta$? Is the support of η compact, and contained in D?
- Surely you are allowed to use some facts from analysis, such as $\mathcal{C}^\infty$ functions with compact support are dense in L^2 functions.

(2) (a) For simplicity, consider only smooth forms (functions, differential, or 2-forms) on $\mathbb{R}^2$ with *compact support.* Note that we can identify a 2-form with a function by

$$f\,\mathrm{d}x \wedge \mathrm{d}y \longleftrightarrow f \ . \qquad (\dagger)$$

With this identification, $\mathrm{d}(\alpha\,\mathrm{d}x + \beta\,\mathrm{d}y) = (\beta_x - \alpha_y)\,\mathrm{d}x \wedge \mathrm{d}y$ is identified with $\beta_x - \alpha_y$. Define a L^2-pairing between functions by

$$(f, g) = \int\!\!\int_{\mathbb{R}^2} f\bar{g}\,\mathrm{d}x\mathrm{d}y \ .$$

Find out the L^2-dual operator of d on functions and differentials. Namely, find out $\delta_0 : \{\text{1-forms}\} \to \{\text{functions}\}$ and $\delta_1 : \{\text{functions}\} \to \{\text{1-forms}\}$ such that

$$(\mathrm{d}f, \omega) = (f, \delta_0\omega) \quad \text{and} \quad (\mathrm{d}\omega, f) = (\omega, \delta_1 f) \ ,$$

for any function f and any differential ω.

(b) If you do part (a) correctly, you shall find the following diagrams

$$\text{function} \underset{\delta_0}{\overset{\mathrm{d}}{\rightleftarrows}} \text{1-form} \underset{\delta_1}{\overset{\mathrm{d}}{\rightleftarrows}} \text{2-forms} \qquad V_0 \underset{A^T}{\overset{A}{\rightleftarrows}} V_1 \underset{JA}{\overset{-A^TJ}{\rightleftarrows}} V_2 \equiv V_0$$

Now, consider the following setting:

V_0 and V_1 are two *finite dimensional* vector space (over $\mathbb{R}$ for simplicity) with inner product. The map A is a linear map from V_0 to V_1, and A^T is the transpose of A by using the inner product. The map J is a linear map from V_1 to itself, which plays the role of the $*$-operator on 1-forms. It satisfies $J^2 = -\mathrm{Id}$ and $\langle Jv, Jw\rangle = \langle v, w\rangle$ for any $v, w \in V_1$.

Suppose that A^TJA is the zero map on V_0. Construct an orthogonal decomposition of V_1 which is parallel to the Hodge decomposition of 1-forms, $L^2(M) = H \oplus E \oplus E^*$.

(3) Consider $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}^2$ as defined in (2) of Homework 1. You may compare this exercise with that one. Let $p_1 = (0,0)$, $p_2 = (\frac{1}{3}, 0)$, $p_3 = (\frac{2}{3}, 0)$, $p_4 = (0, \frac{1}{3})$, $p_5 = (\frac{1}{3}, \frac{1}{3})$, $p_6 = (\frac{2}{3}, \frac{1}{3})$, $p_7 = (0, \frac{2}{3})$, $p_8 = (\frac{1}{3}, \frac{2}{3})$ $p_9 = (\frac{2}{3}, \frac{2}{3})$, and consider $U_j = \{z \in \mathbb{C} \mid |z - p_j| < \frac{1}{3}\}$ for $j \in \{1, 2, \ldots, 9\}$. It is not hard to see that $\{U_j, z\}_{j=1}^9$ consists of a coordinate cover for $\mathbb{C}/\mathbb{Z}^2$.

(a) Check that $\mathrm{d}x$ is a differential on $\mathbb{C}/\mathbb{Z}^2$. (Namely, $\mathrm{d}x$ is a 1-form on each U_j. What happens to the coordinate transition? It is enough to check for U_1 and U_3.)

(b) Is $\mathrm{d}x$ an exact differential? Does there exists a closed curve γ on $\mathbb{C}/\mathbb{Z}^2$ such that $\int_\gamma \mathrm{d}x \neq 0$?

(4) Fix $a > 0$ and $n \in \mathbb{N}$. Consider the following function defined on $D_a = \{z \in \mathbb{C} \mid |z| < a\}$,

$$h(z) = \frac{1}{z^n} + \frac{\bar{z}^n}{a^{2n}} .$$

Check that

$$\lim_{r \to a^-} (*\mathrm{d}h)|_{\partial D_r} = 0 .$$

The restriction of a 1-form on a curve is basically the same procedure for evaluating the line integral: if $\omega = \alpha(x,y)\mathrm{d}x + \beta(x,y)\mathrm{d}y$ and $\gamma : z(t) = x(t) + iy(t)$, then

$$\omega|_\gamma = \big(\alpha(x(t), y(t))\, x'(t) + \beta(x(t), y(t))\, y'(t)\big)\, \mathrm{d}t .$$

You can check that it satisfies the change of variable formula when you change the parametrization of the curve γ, which is indeed the chain rule.