RIEMANN SURFACE
HOMEWORK 1

DUE: TUESDAY, MARCH 8

A compact Riemann surface is a compact, Hausdorff topological space M with the data $\{U_\alpha, z_\alpha\}$ such that

- $\{U_\alpha\}$ is an open cover of M;
- $z_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism onto an open subset of $\mathbb{C}$;
- for any α, β with $U_\alpha \cap U_\beta \neq \varnothing$, $z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ is a biholomorphism (between open subsets of $\mathbb{C}$).

(1) Consider the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with the Riemann surface structure defined during the class. Let $U_0 = \hat{\mathbb{C}} \backslash \{\infty\}$ with coordinate z, and $U_1 = \hat{\mathbb{C}} \backslash \{0\}$ with coordinate w. The transition is $w = 1/z$. Consider the following 1-form

$$\psi = \frac{1}{(1+|z|^2)^2} \mathrm{d}z \quad \text{on} \quad U_0 \ .$$

(a) Work out its expression in terms of (U_1, w).
(b) Consider the (directed) curve γ: $\{z = e^{it} \in \mathbb{C} \mid 0 \le t \le \pi\}$. Evaluate $\int_\gamma \psi$ by using the (U_0, z) coordinate.
(c) Let γ be the same curve as in (b). Evaluate $\int_\gamma \psi$ by using the (U_1, w) coordinate.
(d) Is $\psi + \bar{\psi}$ a harmonic differential?
(e) Let $f(z)$ be a nontrivial rational function on $\mathbb{C}$, i.e. the quotient of two polynomials. It naturally extends as a holomorphic map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. Find out its degree. (You may assume the enumerator and the denominator have no common factor.)

(2) Consider the $\mathbb{Z}^2$-action on $\mathbb{C}$ defined by $(m, n) \cdot z = z + m + ni$. The quotient space $\mathbb{C}/\mathbb{Z}^2$ is a Riemann surface, and the quotient map $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}^2$ is a holomorphic map. Here is one way to construct the local coordinate charts. Let U be the image of $\{z \in \mathbb{C} \mid 0 < \mathrm{Re}\, z < 1\}$ under π, and V be the image $\{z \in \mathbb{C} \mid \frac{1}{2} < \mathrm{Re}\, z < \frac{3}{2}\}$ under π.

The map $z \mapsto \exp(2\pi z)$ descends to a coordinate on U and V. It maps U homeomorphically onto the annulus $\{\xi \in \mathbb{C} \mid 1 < |\xi| < e^{2\pi}\}$, and maps V homeomorphically onto the annulus $\{\eta \in \mathbb{C} \mid e^{\pi} < |\eta| < e^{3\pi}\}$. The image of $U \cap V$ is

$$\{\xi \in \mathbb{C} \mid 1 < |\xi| < e^{\pi} \text{ or } e^{\pi} < |\xi| < e^{2\pi}\} \quad \text{and}$$
$$\{\eta \in \mathbb{C} \mid e^{\pi} < |\eta| < e^{2\pi} \text{ or } e^{2\pi} < |\eta| < e^{3\pi}\}$$

respectively. The transition function is

$$\eta = \begin{cases} e^{2\pi}\xi & \text{when } 1 < |\xi| < e^{\pi} \\ \xi & \text{when } e^{\pi} < |\xi| < e^{2\pi} \end{cases}$$

(a) Consider the 1-form $\varphi = \frac{1}{\xi}\mathrm{d}\xi$ on U. Work out its expression in terms of the η coordinate. Show that it defines a holomorphic differential on $\mathbb{C}/\mathbb{Z}^2$. (More precisely, it extends to define a holomorphic differential.)

(b) Evaluate $\iint_U \varphi \wedge *\bar{\varphi}$.

(c) Write ξ as $x + iy$ (not z). Consider the following 1-form on U:

$$\frac{x\mathrm{d}x + y\mathrm{d}y}{x^2 + y^2} .$$

Find out the star of this 1-form, and show that this 1-form define a harmonic 1-form on $\mathbb{C}/\mathbb{Z}^2$.

(d) With the theory of Mittag-Leffler, the following expression defines a meromorphic function on $\mathbb{C}$.

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \left\{ \frac{1}{(z+m+ni)^2} - \frac{1}{(m+ni)^2} \right\}$$

It is not hard to check that $\wp(z+1) = \wp(z)$ and $\wp(z+i) = \wp(z)$ for any $z \in \mathbb{C}$. It follows that $\wp(z)$ can be regarded as a meromorphic function on $\mathbb{C}/\mathbb{Z}^2$, or equivalently, a holomorphic map from $\mathbb{C}/\mathbb{Z}^2$ to $\hat{\mathbb{C}}$. Find out the degree of this map.

(3) Consider the following change of coordinate on $\mathbb{R}^2$:

$$(x, y) \mapsto (u, v) = (x, y + x) .$$

Introduce the corresponding complex coordinate $z = x + iy$ and $w = u + iv$. Does there exist a nontrivial function on $\mathbb{R}^2$ which is both analytic in the z and w coordinate?