DIFFERENTIAL GEOMETRY II
HOMEWORK 9

DUE: WEDNESDAY, MAY 20

For any $m \in \mathbb{Z}$, consider the 3-manifold X_m defined by

$$X_m = (\mathbb{R}^2 \times \mathbf{S}^1) \cup_{\psi_m} (\mathbb{R}^2 \times \mathbf{S}^1) \ .$$

where the gluing map is defined by

$$\psi_m((r, e^{i\theta}), e^{i\alpha}) = ((\frac{1}{r}, e^{i\theta}), e^{i(\alpha + m\theta)}) \ .$$

(1) When $m = 1$, check that

$$f = \frac{\cos \alpha}{(1 + r^2)^{\frac{1}{2}}}$$

is Morse function with two critical points.

(2) Construct Morse functions for X_m with $m = 0$ and $m = 2$.

(3) Construct a 1-form Θ on X_1 such that

$$\mathrm{d}\Theta = \frac{1}{2\pi} \frac{r \, \mathrm{d}r \wedge \mathrm{d}\theta}{(1 + r^2)^2} \ .$$

Then, compute

$$\int_{X_1} \frac{1}{2\pi} \frac{r \, \mathrm{d}r \wedge \mathrm{d}\theta}{(1 + r^2)^2} \wedge \Theta \ .$$

[*Hint*: Try $\Theta = \mathrm{d}\alpha + a(r)\mathrm{d}\theta$. You have to choose $a(r)$ such that the form is smooth on both charts.]