DIFFERENTIAL GEOMETRY II
HOMEWORK 8

DUE: WEDNESDAY, MAY 13

(1) Let X be a $2n$-dimensional manifold which is compact, connected, oriented with non-empty boundary. Denote by Y the boundary of X. Suppose that $\mathrm{H}^n_{\mathrm{d}}(Y) = 0$. By the Poincaré duality, $\mathrm{H}^{n-1}_{\mathrm{d}}(Y) = 0$ as well.

Fix a diffeomorphism of a neighborhood of Y in X with $Y \times [0, 1)$. For any cohomology class in $\mathrm{H}^n_{\mathrm{d}}(X)$, show that there exists a representative which vanishes on $Y \times [0, \frac{1}{2})$. [*Hint*: Any n-form α on X can be written as $\xi + \mathrm{d}t \wedge \eta$ when restricted on $Y \times [0, 1)$, where ξ is a t-dependent n-form on Y and η is a t-dependent $(n-1)$-form on Y. The closedness of α implies that $\mathrm{d}_Y \xi = 0$ and $\frac{\partial}{\partial t}\xi = \mathrm{d}_Y \eta$.]

(2) Let C and Y be compact, connected, oriented, boundaryless manifolds of dimension $2n$ and $2n-1$, respectively. Suppose that there is an embedding ι of $Y \times (-1, 1)$ into C such that $C \backslash \iota(Y \times \{0\})$ has two components. Denote this two components by X_1 and X_2.

If $\mathrm{H}^n_{\mathrm{d}}(Y) = 0 = \mathrm{H}^{n-1}_{\mathrm{d}}(Y)$, prove that the restriction map

$$\mathrm{H}^n_{\mathrm{d}}(C) \to \mathrm{H}^n_{\mathrm{d}}(X_1) \oplus \mathrm{H}^n_{\mathrm{d}}(X_2)$$

is an isomorphism. [*Hint*: The surjectivity basically follows from (1). You will need $\mathrm{H}^{n-1}_{\mathrm{d}}(Y) = 0$ for the injectivity. You are asked to work out the Meyer–Vietories sequence by hand (if you know what it is), and do not simply invoke it.]

(3) For the following two manifolds of dimension 4, find out b_+ and b_-.

(a) $\mathbf{S}^2 \times \mathbf{S}^2$.

(b) $\mathbf{T}^4 = \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$.

[*Note*: The Künneth formula asserts that $\mathrm{H}^k_{\mathrm{d}}(M_1 \times M_2) = \bigoplus_{i=0}^{k} \big(\mathrm{H}^i_{\mathrm{d}}(M_1) \otimes \mathrm{H}^{k-i}_{\mathrm{d}}(M_2)\big)$.]