DIFFERENTIAL GEOMETRY II
HOMEWORK 7

DUE: WEDNESDAY, MAY 6

(1) Let L be a partial differential operator of order ℓ acting on $\mathbb{C}^m$-valued functions. Write L as

$$L = P_\ell(D) + P_{\ell-1}(D) + \cdots + P_0(D)$$

where $P_j(D) = \sum_{[\alpha]=j} a_\alpha(x)\, D^\alpha$ is homogeneous of oder j. The coefficient functions $a_\alpha(x)$ take value in $\mathbb{M}(m \times m; \mathbb{C})$.

(a) Prove that L is elliptic at x if and only if $L(\varphi^\ell\, \mathfrak{s})(x) \neq 0$ for any $\mathbb{C}^m$-valued function $\mathfrak{s}$ with $\mathfrak{s}(x) \neq 0$ and any smooth real valued function φ with $\varphi(x) = 0$ and $\mathrm{d}\varphi|_x \neq 0$.

In [W], the ellipticity on manifolds is defined by using the above criterion. In most textbooks, the approach is to introduce the *principal symbol*.

Determine whether the following operators are elliptic. Justify your answer.

(b) On $\mathbb{R}^1$ (or $\mathbf{S}^1$), $L = i\,\frac{\partial}{\partial x}$.

(c) On $\mathbb{R}^2$ (or $\mathbf{T}^2$), $L = a\,\frac{\partial}{\partial x} + i\,b\,\frac{\partial}{\partial y}$ where a and b are nonzero real numbers.

(d) The same operator as part (c) on $\mathbb{R}^3$.

(e) On $\mathbb{R}^3$,

$$L = \begin{bmatrix} i\frac{\partial}{\partial z} & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} \end{bmatrix}.$$

(2) Exercise 16 of [W; Chapter 6]. It is about the spectral decomposition induced by Δ. What follows are some hints.

(b) You may need the compactness theorem [W; Theorem 6.6].

(c) Again, use the compactness theorem [W; Theorem 6.6].

(f) You do not have to submit this part. Note that it together with part (c) show that $\lim_{n\to\infty} \lambda_n = \infty$.

(3) Exercise 22 of [W; Chapter 6]. It is about the borderline case of the Sobolev embedding.