DIFFERENTIAL GEOMETRY II
HOMEWORK 6

DUE: WEDNESDAY, APRIL 22

(1) [Toy model of the Hodge Theorem] Let $V^0, V^1, \cdots, V^n$ be *finite* dimensional vector spaces (over $\mathbb{R}$) with inner product. Suppose that there are linear maps $A_j : V^j \to V^{j+1}$ such that $A_{j+1} \circ A_j = 0$. Such a system is usually called a *chain complex*, and is denoted by (V, A).

$$0 \longrightarrow V^0 \xrightarrow{A_0} V^1 \xrightarrow{A_1} \quad \cdots \quad \xrightarrow{A_{j-1}} V^j \xrightarrow{A_j} \quad \cdots \quad \xrightarrow{A_{n-1}} V^n \longrightarrow 0 \ .$$

It is convenient to set A_{-1} and A_n to be the zero map. The *cohomology* of (V, A) is defined to be

$$\mathrm{H}^j(V, A) = \frac{\ker A_j}{\operatorname{im} A_{j-1}} \ .$$

With the help of the inner product, we may consider the adjoint operator of A_j, $A_j^* : V^{j+1} \to V^j$, and introduce the operator $\Delta_j = A_{j-1}A_{j-1}^* + A_j^* A_j$. Let $\mathcal{H}^j = \ker \Delta_j$.

(a) Prove that decomposition: $V^j = \mathcal{H}^j \oplus \Delta_j(V^j)$. As explained in class, it implies that $V^j = \mathcal{H}^j \oplus A_{j-1}A_{j-1}^*(V^j) \oplus A_j^* A_j(V^j) = \mathcal{H}^j \oplus A_{j-1}V^{j-1} \oplus A_j^* V^{j+1}$.

(b) Show that $\mathrm{H}^j(V, A) \cong \mathcal{H}^j$.

(c) Prove that Δ_j maps $A_{j-1}V^{j-1}$ isomorphically onto itself. Also prove the same statement for $A_j^* V^{j+1}$.

(d) Prove that the spectral (eigenspaces) decomposition of Δ_j on $A_{j-1}V^{j-1}$ is equivalent to the spectral decomposition of Δ_{j-1} on $A_{j-1}^* V^j$.

(e) We know that there exists a constant $c > 0$ such that $\|v\| \leq c\|\Delta_j v\|$ for any $v \in (\mathcal{H}^j)^\perp = \Delta_j V^j \subset V^j$. What is the best possible choice of c here?

(f) The *Euler characteristic* of a sequence of vector spaces $\{W^j\}_{j=0}^n$ is defined to be $\chi(W) = \sum_{j=0}^n (-1)^j \dim W^j$. Prove that $\chi(\mathcal{H}) = \chi(V)$.

(g) Does the Poincaré duality hold in this setting? Explain your reason.

(2) Consider the *Green operator* G defined in [W; Definition 6.9]. Show that G is a bounded, self-adjoint operator which takes bounded sequences into sequences with Cauchy subsequences. [*Hint*: It is not surprising that you have to invoke Theorem 6.6.]

Also, read Proposition 6.10. You don't have to submit this part.

(3) Exercise 7 of [W; Chapter 6]. It is the L^2-norm on T^2, up to a factor of 2π. Roughly speaking, it says that the Laplace dominates all the second order derivatives. [*Hint*: Perform integration by parts.]

(4) Exercise 8 of [W; Chapter 6]. It asks you to work out the Rellich lemma in the simplest case by hand.