DIFFERENTIAL GEOMETRY II
HOMEWORK 5

DUE: WEDNESDAY, APRIL 15

(1) Let V be a finite dimensional vector space with an inner product and a chosen orientation. With the inner product and the star operator defined in class, prove the following properties.

  (a) $\langle w_1 \wedge \cdots \wedge w_k, v_1 \wedge \cdots \wedge v_k \rangle = \det\langle w_i, v_j \rangle$ for any $w_1 \wedge \cdots \wedge w_k, v_1 \wedge \cdots \wedge v_k \in \Lambda^k V$. The pairing $\langle w_i, v_j \rangle$ is the given inner product on V.

  (b) $\langle w, v \rangle = *(w \wedge *v)$ for any $w, v \in \Lambda^k V$.

(2) On an oriented, Riemannian manifold, check that $*\Delta = \Delta *$.

(3) On $\mathbb{R}^n$ with the standard metric, show that the Laplace–Beltrami operator acting on a k-form

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha_I \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$$

reads

$$\Delta\alpha = -\sum_{i_1 < \cdots < i_k} \left(\sum_{j=1}^{n} \frac{\partial^2 \alpha_I}{\partial x_j^2} \right) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \ .$$

(4) On an oriented, Riemannian manifold M, prove that $\Delta f = -\operatorname{div} \nabla f$ for any $f \in \mathcal{C}^\infty(M)$. The right hand side is defined in (5) of Homework 3.

(5) On an oriented, Riemannian manifold M^n, choose a local coordinate $\{x^j\}_{j=1}^n$ such that $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ is positively oriented. Let g_{ij} be the coefficients of the Riemannian metric. It is not hard to see that $*1 = \sqrt{\det g_{ij}}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ and $*(\sqrt{\det g_{ij}}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = 1$.

  (a) Find out $*\mathrm{d}x^j$. In any case, it shall be a linear combination of $\{\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_{n-1}}\}_{i_1 < \cdots < i_{n-1}}$, and (1.b) would determine the coefficients. Also, note that $\langle \mathrm{d}x^i, \mathrm{d}x^j \rangle = g^{ij}$, the inverse of g_{ij}. [*Hint*: It is convenient to introduce the $(n-1)$-form $\xi_i = (-1)^{i-1}\, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n = \iota_{\partial_i}(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n)$.]

  (b) Work out the expression of $\Delta f = \delta \mathrm{d} f$ in the local coordinate system. From this expression, it shall be easier to see that $\Delta = -\operatorname{div} \nabla$ is a self-adjoint operator.