DIFFERENTIAL GEOMETRY II
HOMEWORK 4

DUE: WEDNESDAY, APRIL 8

(1) Exercise 3 of Chapter 5 in [dC, p.119] (about the non-existence of conjugate points).

(2) Exercise 5 of Chapter 10 in [dC, p.238] (the Sturm comparison theorem).

The rest of this homework set is about the geometric properties of a submanifold. These properties are mainly about how the submanifold sits inside the ambient manifold, but not just about the metric on the submanifolds. In this regards, they are usually referred as *extrinsic* properties of the submanifolds. The reference is [dC, chapter 6] and [CE, section 1.7].

Let $(\bar{M}^{n+k}, \bar{g})$ be a Riemannian manifold, and denote its Levi-Civita connection by $\bar{\nabla}$. Suppose that M^n is a submanifold of $\bar{M}$. Then $\bar{g}$ restricts to a Riemannian metric g on M. For any $p \in M$,

$$T_p\bar{M} = T_pM \oplus (T_pM)^{\perp} \ .$$

The second summand is the fiber of the normal bundle of M in $\bar{M}$.

In what follows, $X, Y, Z, \ldots$ are local vector fields on M, and $\bar{X}, \bar{Y}, \bar{Z}, \ldots$ are their local extensions to $\bar{M}$. The local normal vector fields (sections of $(TM)^{\perp} \subset T\bar{M}$) will be denoted by $\eta, \zeta, \ldots$.

(3) Check that the following map is *symmetric*, and *functional linear*:

$$\mathrm{I\!I}(X, Y) = \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y$$

According to (4) of homework 1, it takes value in the normal bundle. It is called the *second fundamental form*, or the *shape tensor* of M in $\bar{M}$.

Given a normal vector (field) η, the quantity $\mathrm{I\!I}_\eta(X, Y) = \bar{g}(\mathrm{I\!I}(X, Y), \eta)$ is called the second fundamental form of M in $\bar{M}$ along η. You can consult [dC, p.129] for the case when $k = 1$, which is closely related to the story you learnt in undergraduate geometry.

In a way the second fundamental form encodes the curvature of the ambient manifold $\bar{M}$. You can image that the curvatures of M and $\bar{M}$ are related to each other by the second fundamental form. Prove the following formulae.

(4) Th Gauss equation:

$$\bar{g}(\bar{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + \bar{g}(\mathrm{I\!I}(X, Z), \mathrm{I\!I}(Y, W)) - \bar{g}(\mathrm{I\!I}(X, W), \mathrm{I\!I}(Y, Z)) \ .$$

(5) The Codazzi equation:

$$\bar{g}(\bar{R}(X,Y)Z,\eta) = (\bar{\nabla}_X \mathrm{II})(Y,Z,\eta) - (\bar{\nabla}_Y \mathrm{II})(X,Z,\eta) \ .$$

Here, II is considered as a tensor defined by $\mathrm{II}(X,Y,\eta) = \bar{g}(\mathrm{II}(X,Y),\eta)$. Thus,

$$(\bar{\nabla}_X \mathrm{II})(Y,Z,\eta) = X\left(\mathrm{II}_\eta(Y,Z)\right) - \mathrm{II}_\eta(\nabla_X Y, Z) - \mathrm{II}_\eta(Y,\nabla_X Z) - \bar{g}(\mathrm{II}(Y,Z),\bar{\nabla}_X\eta) \ .$$

If the second fundamental form of M vanishes, M is called a *totally geodesic* submanifold. It implies that any geodesic of (M,g) is also a geodesic in $(\bar{M},\bar{g})$.

Another notion is the *mean curvature vector* of M, which is defined to be the trace (with respect to g) of the second fundamental form. Namely,

$$H = \sum_{j=1}^{n} \mathrm{II}(e_j,e_j)$$

where $\{e_j\}$ is a local orthonormal frame. The mean curvature vector is a section of the normal bundle. If H vanishes, the submanifold is called a *minimal* submanifold. Clearly a totally geodesic submanifold is minimal.

(6) Bonus Suppose that M is orientable. Consider an orientation-compatible local coordinate system $\{x^j\}$. The *volume form* of (M,g) is $\sqrt{\det(g_{ij})}\,\mathrm{d}x^1\wedge\cdots\mathrm{d}x^n$. It is not hard to show that such an element is independent of the choice of coordinates. If M is also compact, its volume is defined to be $\int_M \sqrt{\det(g_{ij})}\,\mathrm{d}x^1\wedge\cdots\mathrm{d}x^n$.

Let η be a normal vector field of M. Consider the deformation of M along η: $F_s(p) = \overline{\exp}_p(s\,\eta(p))$. In other words, $F_s : M\times(-\epsilon,\epsilon)$ is a one-parameter family of embeddings of M into $\bar{M}$ with F_0 to be the original embedding and $\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}F_s = \eta$. The main purpose of this exercise is to compute $\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\mathrm{vol}(F_s(M))$. In terms of local coordinate, it is

$$\int_M \left(\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\sqrt{\det(g_{ij}(s))}\right)\,\mathrm{d}x^1\wedge\cdots\mathrm{d}x^n \ .$$

Due to (1) of Homework 2 last semester, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\sqrt{\det(g_{ij}(s))} = \frac{1}{2}g^{ij}\left(\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}g_{ij}(s)\right)\sqrt{\det(g_{ij})}$$

Prove that

$$\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}g_{ij}(s) = -2\bar{g}\left(\eta,\bar{\nabla}_{F_{0*}(\frac{\partial}{\partial x^i})}F_{0*}(\frac{\partial}{\partial x^j})\right) \ ,$$

and use it to conclude that $\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\mathrm{vol}(F_s(M)) = -\int_M \bar{g}(\eta,H)\sqrt{\det(g_{ij})}\,\mathrm{d}x^1\wedge\cdots\mathrm{d}x^n$. Roughly speaking, the mean curvature vector is the negative gradient of the volume functional. Minimal submanifolds are the critical points of the volume functional. [*Hint*: It uses similar properties in the discussion of the Jacobi field equation.]