DIFFERENTIAL GEOMETRY II
HOMEWORK 3

DUE: WEDNESDAY, MARCH 25

(1) By using [CE, (1.10)], it is not hard to check that a metric g has constant sectional curvature K if and only if

$$g(R(x,y)z,w) = K\,(g(x,w)\,g(y,z) - g(x,z)\,g(y,w))\ .$$

In terms of a local orthonormal frame $\{e_j\}$, the condition reads $R_{ijij} = K = -R_{ijji}$ for any $i \neq j$, and other curvature components vanishes. Here, $R_{ijk\ell} = g(R(e_k, e_\ell)e_j, e_i)$. Check the following metrics has constant sectional curvature, and finds out the sectional curvature. (c is a positive constant.)

(a) $g = \left(\frac{2\,c}{1+|x|^2}\right)^2 \sum_j (\mathrm{d}x^j)^2$ on $\mathbb{R}^n$.

(b) $g = \left(\frac{2\,c}{1-|x|^2}\right)^2 \sum_j (\mathrm{d}x^j)$ on B^n.

[*Remark*: In terms of an orthonormal frame, $\Omega^i_j = \frac{1}{2}\,R_{ijk\ell}\,\omega^k \wedge \omega^\ell$.]

(2) Let (M,g) be a Riemannian manifold. Suppose that $\gamma : [0,\ell] \to M$ is a geodesic.

(a) Denote the tangent vector field of γ by T. Check that T and $t\,T$ are both Jacobi fields along γ. These two vector fields correspond to uninteresting geodesic variations of γ.

(b) For any Jacobi field J along γ, prove that $g(J,T)$ is a affine function in t, i.e. $g(J,T) = a_0 + a_1\,t$. Also determine the coefficients a_0 and a_1.

(c) Due to part (a) and (b), we may focus on those Jacobi fields J obeying $g(J,T) \equiv 0$. If the manifold (M,g) has constant sectional curvature, work out the general expression of a Jacobi field along γ (in terms of a parallel, orthonormal frame $\{E_j\}$ along γ). And determine when a conjugate point occur along γ.

(3) Exercise 7 of Chapter 3 in [dC, p.83] (about the geodesic frame). [*Hint*: Consider the exponential map at p. Choose an orthonormal basis $\{e_j\}$ for T_pM, and parallel transport them along radial geodesics.]

(4) Exercise 7 of Chapter 4 in [dC, p.106] (about the second Bianchi identity). [*Remark*: In [dC], the notation $(\nabla R)(U,V,W,X,Y)$ means $(\nabla_Y R)(U,V,W,X)$, and thus is equal to

$$Y(g(R(U,V)W,X)) - g(R(\nabla_Y U,V)W,X)) - g(R(U,\nabla_Y V)W,X)) \\ - g(R(U,V)(\nabla_Y W),X)) - g(R(U,V)W,\nabla_Y W))\ .$$

In most other books, $(\nabla R)(U, V, W, X, Y)$ means $(\nabla_U R)(V, W, X, Y)$.]

(5) {From [dC, exercise 8 & 9 of ch.3]} On a Riemannian manifold (M, g), introduce the following notions.

- For a smooth function $f : M \to \mathbb{R}$, its *gradient*, ∇f, is the vector field defined by

$$g(\nabla f, X) = X(f) \quad \text{for any vector field } X \ .$$

- For a vector field V, its *divergence*, $\operatorname{div} V$, is the smooth function defined to be the trace of the linear map $X \mapsto \nabla_X V$.
- For a smooth function f, its *Laplacian*, Δf, is the smooth function $\operatorname{div}(\nabla f)$.

You are asked to work out these notions in terms of a local coordinate system $\{x^j\}$. Denote the coefficients of the metric by g_{ij}.

(a) Write ∇f as $a^j \frac{\partial}{\partial x^j}$; find out a^j.

(b) For a vector field $V = v^j \frac{\partial}{\partial x^j}$; find out $\operatorname{div} V$.

(c) Combine part (a) and (b) to write down the expression of Δf in local coordinate.