DIFFERENTIAL GEOMETRY II
HOMEWORK 2

DUE: WEDNESDAY, MARCH 18

(1) Let G be a Lie group, and g be a *left-invariant* metric on G. Suppose that X, Y, Z, W are *left-invariant* vector fields on G. Prove the following formulae.

(a) The Levi-Civita connection is given by

$$g(\nabla_X Y, Z) = \frac{1}{2}\left(g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)\right) \ .$$

(b) The Riemann curvature tensor is given by

$$g(R(X,Y)Z,W) = g(\nabla_X Z, \nabla_Y W) - g(\nabla_Y Z, \nabla_X W) - g(\nabla_{[X,Y]}Z, W) \ .$$

(2) Let (M, g) be a Riemannian manifold, and V be a vector field on M. It is called a *Killing vector field* if it preserves the metric, i.e. $L_V g = 0$. Equivalently, the one-parameter family of diffeomorphism generated by V is an isometry.

(a) Show that V is a Killing vector field if and only if

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) = 0$$

for any X, Y. [*Hint*: Remember that $(L_V g)(X, Y)$ is defined by the Leibniz rule, and is equal to $V(g(X,Y)) - g(L_V X, Y) - g(X, L_V Y)$.]

(b) Let $\gamma(t) : (-\epsilon, \epsilon) \to M$ be a geodesic. Prove that $g(V, \gamma')$ is a constant (along γ) if V is a Killing vector field.

(c) Let $\gamma(t) : (-\epsilon, \epsilon) \to M$ be a geodesic. Prove that the restriction of V along γ is a Jacobi field if V is a Killing vector field. [*Hint*: Instead of checking the Jacobi field equation, there is a more geometric reason.]

(3) Let (M, g) be a Riemannian manifold whose sectional curvature is greater than 1. Suppose that $\gamma(t) : [0,1] \to M$ is a smooth curve with unit speed, and $V(t)$ is a nonzero parallel vector field along γ. For $s \in (-\epsilon, \epsilon)$, consider the curve $\gamma_s(t) : [0,1] \to M$ defined by $\exp_{\gamma(t)}(sV(t))$. For s sufficient small, compare the length of $\gamma_s(t)$ and $\gamma(t)$. Explain your answer. [*Hint*: For any $t_0 \in [0,1]$. Consider the vector field $\frac{\partial}{\partial t}\big|_{t=t_0}\gamma_s$ along the s-curve $\gamma_s(t_0)$.]

(4) Given a covariant derivative ∇ on a vector bundle E, it naturally induces a covariant derivative on its algebraic constructions: $\otimes^k E, \mathrm{End}(E,E)$, etc. The basic principle is the Leibniz rule.

Let (M, g) be a Riemannian manifold, and let ∇ be the Levi-Civita connection.

(a) Show that g is a *parallel* section of $\mathrm{Hom}(TM \otimes TM, TM)$. It is usually denoted by $\nabla g = 0$. [*Hint*: The hint of (2.a) would also help you here.]

(b) Let $\{e_j\}$ is a local orthonormal frame. Denote by ω_j^k the coefficient 1-forms of the Levi-Civita connection in terms of $\{e_j\}$, i.e. $\nabla e_j = \omega_j^k\, e_k$. Let $\{\omega^j\}$ be the dual coframe of $\{e_j\}$. Find out $\nabla \omega^j$. Note that $\nabla \omega^j$ is a section of $T^*M \otimes T^*M$. [*Remark*: In fact, the skew-symmetrization of $\nabla \omega^j$ is $\mathrm{d}\omega^j$.]

(c) (continued from part (a) and (b)) Then, the metric g is $\sum_j \omega^j \otimes \omega^j$. Use your result of part (b) to re-do part (a).