DIFFERENTIAL GEOMETRY II
HOMEWORK 1

DUE: WEDNESDAY, MARCH 11

(1) Consider the Poincaré disk model of the hyperbolic geometry:

$$g = \frac{4}{\left(1 - \sum_{j=1}^{n}(x^j)^2\right)^2}\sum_{j=1}^{n} \mathrm{d}x^j \otimes \mathrm{d}x^j$$

on $D = \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid \sum_{j=1}^{n}(x^j)^2 < 1\}$.

(a) Calculate its Levi-Civita connection.

(b) Describe its Riemann curvature tensor, as a section of $\Lambda^2 T^*D \otimes \mathrm{End}(TD)$.

You are suggested to do it by the method of moving frame.

(2) Consider the metric

$$g = A(r)^2\,\mathrm{d}r \otimes \mathrm{d}r + r^2\,\mathrm{d}\phi \otimes \mathrm{d}\phi + r^2 \sin^2\phi\,\mathrm{d}\theta \otimes \mathrm{d}\theta$$

on $M = I \times \mathbf{S}^2$. Here, r is the coordinate on the interval I, and (ϕ, θ) is the spherical coordinate on $\mathbf{S}^2$.

(a) Calculate its Levi-Civita connection.

(b) Describe its Riemann curvature tensor, as a section of $\Lambda^2 T^*M \otimes \mathrm{End}(TM)$.

(3) {From [dC, p.46]} Prove that the isometries of $\mathbf{S}^n \subset \mathbb{R}^{n+1}$ with the induced metric, are the restrictions to $\mathbf{S}^n$ of $\mathrm{O}(n+1)$. [*Hint*: Let $f : \mathbf{S}^n \to \mathbf{S}^n$ be an isometry. Consider $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ which sends $\mathbf{x} \neq 0$ to $|\mathbf{x}|\, f(\mathbf{x}/|\mathbf{x}|)$ and sends 0 to 0.]

(4) {From [dC, p.57]} Let $(\bar{M}, \bar{g})$ be a Riemannian manifold, and denote its Levi-Civita connection by $\bar{\nabla}$. Suppose that M is a submanifold of $\bar{M}$. Then $\bar{g}$ restricts to a Riemannian metric g on M. For any two (locally defined) smooth vector fields X, Y on M, consider the following construction.

First, extend X, Y to smooth vector fields on an open set $U \subset \bar{M}$, and denote the extension by $\bar{X}, \bar{Y}$. For any $p \in M \cap U$, define $(\nabla_X Y)|_p = \mathrm{pr}(\bar{\nabla}_{\bar{X}}\bar{Y})|_p$. Here, pr is the projection from $T_p\bar{M} = T_pM \oplus N_p$ onto the first summand, where N_p is the normal bundle of M in $\bar{M}$ at p. Prove that ∇ is the Levi-Civita connection of (M, g). [*Hint*: The Levi-Civita connection is characterized uniquely by two properties.]

(5) Given a Riemannian metric $g = \sum_{i,j} g_{ij}(\mathbf{x})\, \mathrm{d}x^i \otimes \mathrm{d}x^j$, consider its Riemann curvature tensor

$$R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell})\frac{\partial}{\partial x^i} = \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^\ell}} \frac{\partial}{\partial x^i} - \nabla_{\frac{\partial}{\partial x^\ell}} \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} - \nabla_{[\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}]} \frac{\partial}{\partial x^i}$$
$$= \sum_j R^j_{ik\ell} \frac{\partial}{\partial x^j} \ .$$

Work out the expression of $R^j_{ik\ell}$ in terms of the Christoffel symbols and their derivatives.