NOTE ON MARCH 25

DIFFERENTIAL GEOMETRY II

Tensor Calculus

This note is a brief introduction to classical tensor calculus, which is basically a notation for doing local calculation of covariant derivatives. You can consult [A] for a complete treatment of this method. We will demonstrate that the restriction of a Killing vector field is a Jacobi field.

Basic convention. Let (M, g) be a Riemannian manifold, and ∇ be its Levi-Civita connection. Choose a local coordinate $\{x^j\}$. Let g_{ij} be the coefficient functions of the Riemannian metric. For simplicity, we abbreviate $\frac{\partial}{\partial x^j}$ as ∂_j. Given a vector field $V = v^j \partial_j$, denote the components of its covariant derivative by $v^j_{\ ;k}$. Namely,

$$\nabla V = v^j_{\ ;k}\, \mathrm{d}x^k \otimes \partial_j \qquad \text{and} \quad v^j_{\ ;k} = \frac{\partial v^j}{\partial x^k} + \Gamma^j_{ki}\, v^i \ .$$

Curvature. The components of the second order covariant derivative of V are denoted by $v^j_{\ ;k\ell}$. That is to say, $v^j_{\ ;k\ell}$ is the ∂_j-component of the following vector field:

$$\begin{aligned}(\nabla^2 V)(\partial_\ell, \partial_k) &= \left(\nabla_{\partial_\ell}(\nabla V)\right)(\partial_k) \\ &= \nabla_{\partial_\ell}\nabla_{\partial_k} V - \nabla_{\nabla_{\partial_\ell}\partial_k} V \ .\end{aligned}$$

It is not hard to verify that

$$v^j_{\ ;k\ell} = \frac{\partial v^j_{\ ;k}}{\partial x^\ell} + \Gamma^j_{\ell i}\, v^i_{\ ;k} - \Gamma^i_{\ell k}\, v^j_{\ ;i} \ .$$

Let R^ℓ_{kij} be the ∂_ℓ-component of $R(\partial_i, \partial_j)\partial_k$; namely,

$$R(\partial_i, \partial_j)\partial_k = R^\ell_{kij}\, \partial_\ell \ .$$

We have the following commuting covariant derivatives formula:

$$v^j_{\ ;k\ell} - v^j_{\ ;\ell k} = R^j_{i\ell k}\, v^i \ .$$

Another commonly used notation is $R_{\ell kij}$. It is

$$R_{\ell kij} = g_{\ell m}\, R^m_{kij} = g(R(\partial_i, \partial_j)\partial_k, \partial_\ell) \ .$$

Dual to the cotangent bundle. The metric g induces an isomorphism between TM and T^*M. Applying it to the vector field V gives a 1-form α_V. More precisely, α_V is defined by $\alpha_V(W) = g(V,W)$ for any vector field W. The local expression of α_V is $g_{ij}v^j\,\mathrm{d}x^i$. Let $v_i = g_{ij}v^j$. Note that $v^j = g^{ji}v_i$. Recall that the covariant derivative of α_V is defined by

$$(\nabla_U\alpha_V)(W) = U(\alpha_V(W)) - \alpha_V(\nabla_U W)\ .$$

Denote the components of the covariant derivative of α_V by $v_{i;j}$; namely,

$$\nabla\alpha_V = v_{i;j}\,\mathrm{d}x^j\otimes\mathrm{d}x^i \qquad \text{or} \qquad \nabla_{u^j\partial_j}\alpha_V = v_{i;j}\,u^j\,\mathrm{d}x^i\ .$$

You can check the following relations.

$$\begin{aligned} v_{i;j} = (g_{ik}\,v^k)_{;j} &= \frac{\partial v_i}{\partial x^j} - \Gamma^k_{ij}\,v_k \\ &= g_{ik}\,v^k_{\ ;j}\ , \\ v_{j;k\ell} - v_{j;\ell k} &= R^i_{jk\ell}\,v_i\ . \end{aligned}$$

Killing vector field. Suppose that V is a Killing vector field. By (2.a) of Homework 2, it is equivalent to

$$\begin{aligned} & g(\nabla_{\partial_i}V,\partial_j) + g(\nabla_{\partial_j}V,\partial_i) = 0 \\ \Leftrightarrow\quad & g_{kj}\,v^k_{\ ;i} + g_{ki}\,v^k_{\ ;j} = 0 \\ \Leftrightarrow\quad & v_{j;i} + v_{i;j} = 0\ . \end{aligned}$$

It follows that

$$\begin{aligned} v_{j;ik} &= -v_{i;jk} & &= -R^\ell_{ijk}\,v_\ell + R^\ell_{kij}\,v_\ell + v_{k;ji} \\ &= -R^\ell_{ijk}\,v_\ell - v_{i;kj} & &= -R^\ell_{ijk}\,v_\ell + R^\ell_{kij}\,v_\ell - v_{j;ki} \\ &= -R^\ell_{ijk}\,v_\ell + v_{k;ij} & &= -R^\ell_{ijk}\,v_\ell + R^\ell_{kij}\,v_\ell - R^\ell_{jki}\,v_\ell - v_{j;ik}\ . \end{aligned}$$

By the Bianchi identity,

$$v_{j;ik} = R^\ell_{kij}\,v_\ell\ .$$

Suppose that $\gamma(t) = (x^1(t),\dots,x^n(t))$ is a geodesic. Denote it tangent vector field by T. We would like to compute $\nabla_T\nabla_T V$. Since $\nabla_T T = 0$, $\nabla_T\nabla_T V = (\nabla^2 V)(T,T)$, and is equal to $v^\ell_{\ ;ik}\,\dot{x}^i\dot{x}^k\,\partial_\ell$. We compute

$$\begin{aligned} v^\ell_{\ ;ik}\,\dot{x}^i\dot{x}^k &= g^{\ell j}\,v_{j;ik}\,\dot{x}^i\dot{x}^k & &= g^{\ell j}\,R_{qkij}\,v^q\,\dot{x}^i\dot{x}^k \\ &= g^{\ell j}\,R^p_{kij}\,v_p\,\dot{x}^i\dot{x}^k & &= g^{\ell j}\,R_{jikq}\,v^q\,\dot{x}^i\dot{x}^k \\ &= g^{\ell j}\,g^{pq}\,R_{qkij}\,v_p\,\dot{x}^i\dot{x}^k & &= R^\ell_{ikq}\,v^q\,\dot{x}^i\dot{x}^k\ , \end{aligned}$$

which is exactly the ∂_ℓ-component of $R(T,V)T$. Thus, $\nabla_T\nabla_T V = R(T,V)T$, and the restriction of a Killing field on a geodesic is a Jacobi field.

References

[A] T. Aubin, *A course in differential geometry*, Graduate Studies in Mathematics, vol. 27, American Mathematical Society, Providence, RI, 2001.