NOTE ON MARCH 4

DIFFERENTIAL GEOMETRY II

The method of moving frame. Let (M, g) be a Riemannian manifold. Choose a local orthonormal frame $\{\mathfrak{e}_1, \cdots, \mathfrak{e}_n\}$, and let $\{\omega^1, \cdots, \omega^n\}$ be the dual coframe. Let ω_j^i be the connection 1-form of the Levi-Civita connection of g in terms of $\{\mathfrak{e}_j\}$. Namely,

$$\nabla\mathfrak{e}_j = \omega_j^i\,\mathfrak{e}_i\ .$$

Since $\{\mathfrak{e}_j\}$ is orthonormal,

$$\begin{aligned} 0 &= \mathrm{d}\left(g(\mathfrak{e}_i, \mathfrak{e}_j)\right) \\ &= g(\nabla\mathfrak{e}_i, \mathfrak{e}_j) + g(\mathfrak{e}_i, \nabla\mathfrak{e}_j) \\ &= \omega_i^j + \omega_j^i\ . \end{aligned} \tag{1}$$

Thus, ω_i^j is skew-symmetric. According to the torsion free condition,

$$\begin{aligned} & \nabla_{\mathfrak{e}_i}\mathfrak{e}_j - \nabla_{\mathfrak{e}_j}\mathfrak{e}_i = [\mathfrak{e}_i, \mathfrak{e}_j] \\ \Rightarrow\quad & \omega_j^k(\mathfrak{e}_i) - \omega_i^k(\mathfrak{e}_j) = \omega^k\left([\mathfrak{e}_i, \mathfrak{e}_j]\right) \\ &\qquad\qquad\qquad\qquad = -(\mathrm{d}\omega^k)(\mathfrak{e}_i, \mathfrak{e}_j)\ . \end{aligned}$$

It is not hard to see that this equation is equivalent to

$$-\mathrm{d}\omega^k = \omega_\ell^k \wedge \omega^\ell\ , \tag{2}$$

which is called the structure equation.

The connection 1-forms ω_i^j is determined uniquely from the structure equation and the skew-symmetric property;

$$\omega_i^j(\mathfrak{e}_k) = \frac{1}{2}\Big((\mathrm{d}\omega^j)(\mathfrak{e}_i, \mathfrak{e}_k) - (\mathrm{d}\omega^i)(\mathfrak{e}_j, \mathfrak{e}_k) + (\mathrm{d}\omega^k)(\mathfrak{e}_i, \mathfrak{e}_j)\Big)\ .$$

Riemann curvature tensor. What follows is a brief review on the construction of the curvature (see [T, §12.4]). Let $V = v^i\mathfrak{e}_i$. Then,

$$\begin{aligned} \mathrm{d}_\nabla V &= \nabla V \\ &= \mathfrak{e}_i \cdot (\mathrm{d}v^i + \omega_j^i\, v^j) \\ \mathrm{d}_\nabla^2 V &= (\nabla\mathfrak{e}_i) \wedge (\mathrm{d}v^i + \omega_j^i\, v^j) + \mathfrak{e}_i \cdot \mathrm{d}(\mathrm{d}v^i + \omega_j^i\, v^j) \\ &= \mathfrak{e}_i \cdot (\mathrm{d}\omega_j^i + \omega_k^i \wedge \omega_j^k)\, v^j\ . \end{aligned}$$

The endomorphism-valued 2-form

$$\begin{aligned}\Omega^i_j &= \mathrm{d}\omega^i_j + \omega^i_k \wedge \omega^k_j \\ &= \mathrm{d}\omega^i_j - \omega^k_j \wedge \omega^i_k\end{aligned} \tag{3}$$

is the curvature.

For any two vector fields X and Y,

$$\begin{aligned}\big(F^2_\nabla(X,Y)\big)(V) &= \mathfrak{e}_i \cdot \Big((\mathrm{d}\omega^i_j)(X,Y) + \omega^i_k(X)\omega^k_j(Y) - \omega^i_k(Y)\omega^k_j(X)\Big)\, v^j \\ &= \mathfrak{e}_i \cdot \Big(X(\omega^i_j(Y)) + \omega^i_k(X)\omega^k_j(Y) \\ &\qquad - Y(\omega^i_j(X)) - \omega^i_k(Y)\omega^k_j(X) - \omega^i_j\big([X,Y]\big)\Big)v^j \ .\end{aligned}$$

It is related to the following covariant derivatives.

$$\begin{aligned}\nabla_X\nabla_Y V &= \nabla_X \left(Y(v^i) + \omega^i_j(Y)v^j\right)\mathfrak{e}_i \\ &= \Big(X(Y(v^i)) + X(\omega^i_j(Y))v^j + \omega^i_j(Y)X(v^j) + \omega^i_k(X)\left(Y(v^k) + \omega^k_j(Y)v^j\right)\Big)\,\mathfrak{e}_i \ , \\ -\nabla_Y\nabla_X V &= -\nabla_Y\left(X(v^i) + \omega^i_j(X)v^j\right)\mathfrak{e}_i \\ &= \Big(-Y(X(v^i)) - Y(\omega^i_j(X))v^j - \omega^i_j(X)Y(v^j) - \omega^i_k(Y)\left(X(v^k) + \omega^k_j(X)v^j\right)\Big)\,\mathfrak{e}_i \ , \\ -\nabla_{[X,Y]}V &= \left(-[X,Y](v^i) - \omega^i_j([X,Y])v^j\right)\mathfrak{e}_i \ .\end{aligned}$$

It follows that

$$\big(F^2_\nabla(X,Y)\big)(V) = \nabla_X\nabla_Y V - \nabla_Y\nabla_X V - \nabla_{[X,Y]}V \ . \tag{4}$$

This is usually denoted by $R(X,Y)V$, and is called the Riemann curvature tensor.