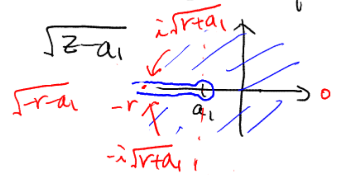
Given $a_1, a_2, \cdots, a_{2n} \in \mathbb{R}$, $\quad a_1 < a_2 < \cdots < a_{2n}$

Consider $f(z) = \sqrt{(z-a_1)(z-a_2)\cdots(z-a_{2n})} = (g(z))^{\frac{1}{2}}$

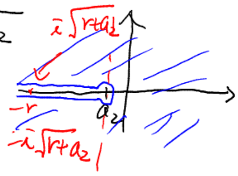
Domain of $f(z)$?

1° Start with the simplest case $\sqrt{(z-a_1)(z-a_2)}$

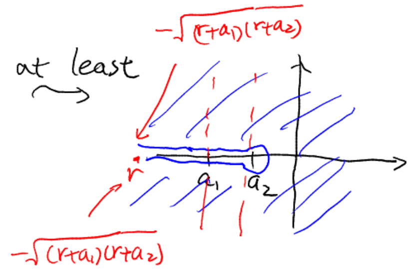
for $\sqrt{z-a_1}$



for $\sqrt{z-a_2}$

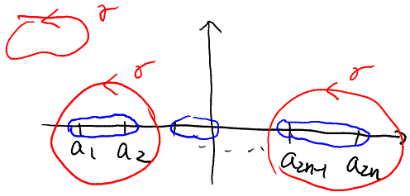


at least $\rightsquigarrow$



Examine the range $\Rightarrow$ $\sqrt{(z-a_1)(z-a_2)}$ is a well-defined analytic function on $\mathbb{C} \setminus [a_1, a_2]$

2° Inductively, $\sqrt{(z-a_1)(z-a_2)\cdots(z-a_{2n})}$ is well-defined on $\mathbb{C} \setminus \{[a_1,a_2] \cup \cdots \cup [a_{2n-1}, a_{2n}]\}$



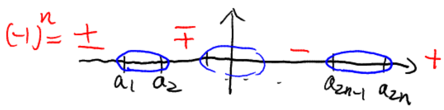
Or γ: any closed loop in $\mathbb{C} \setminus \{[a_1,a_2] \cup \cdots \cup [a_{2n-1}, a_{2n}]\}$

$$\frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz$$

$$\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{z-a_1} + \cdots + \frac{1}{z-a_{2n}}\right) dz = \sum_{j=1}^{2n} n(\gamma; a_j) \text{ is even}$$

$\rightsquigarrow$ $\exp\left(\frac{1}{2}\int_{z_0}^{z} \frac{g'(w)}{g(w)}\,dw\right)$ is a well-defined analytic function

3° Examine the value on $\mathbb{R}$



$f(z)$ "$= \sqrt{g(z)}$"

Wait! We can get another square root of $g(z)$ by $\times -1$

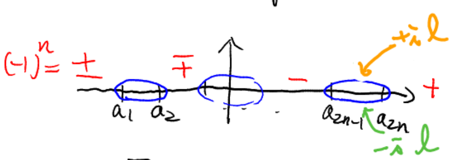
$f(z)$ "$= -\sqrt{g(z)}$"

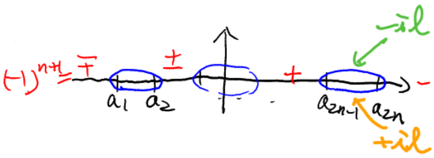
rmk the branch of $(g(z))^{\frac{1}{2}}$ is determined by its value at $x \in \mathbb{R}$, $x > a_{2n}$

4° In other words, $(g(z))^{\frac{1}{2}}$ is a multi-valued function (two)

Why don't we just consider it on two copies of $\mathbb{C} \setminus \{[a_1,a_2] \cup \cdots \cup [a_{2n-1}, a_{2n}]\}$?

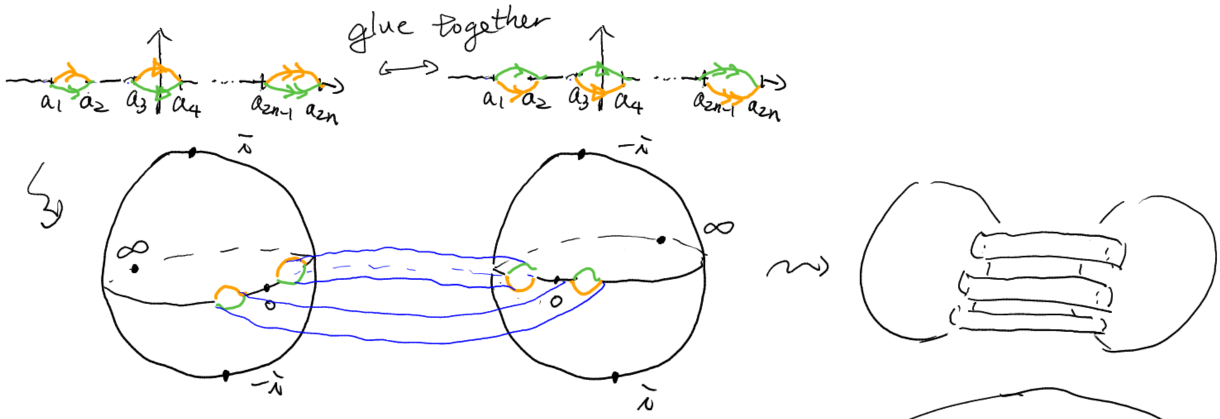
$$l = \sqrt{a_{2n} - r}\sqrt{\prod_{j=1}^{2n-1}(r - a_j)}$$





Examine what happens near $[a_{2j-1}, a_{2j}]$.

We shall think the interval as a "tunnel" via which we can travel between these two copies of $\mathbb{C}$



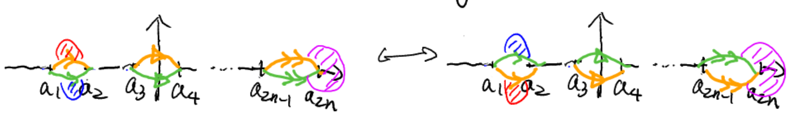
$\rightsquigarrow$ a genus $(n-1)$ surface Σ

5° $f(z) = (g(z))^{\frac{1}{2}}$ is a well-defined (single-valued) meromorphic function on Σ

What does holomorphic (analytic) / meromorphic mean?

For $\mathbb{C} \setminus \{[a_1,a_2] \cup \cdots \cup [a_{2n-1},a_{2n}]\}$: two copies, it is clear

For ⊘ and ⊘, also analytic

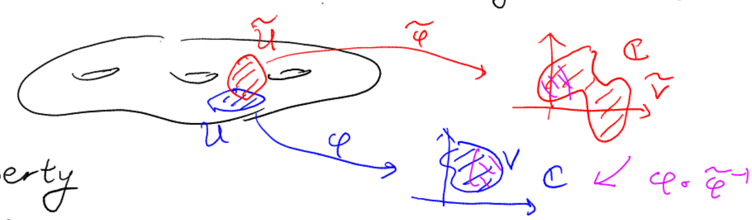


Near a_j? ← a_j does NOT have two copies.

$\Rightarrow f(z) = g(z)^{\frac{1}{2}}$ "=" $w\,h(w)$ ↳ nonzero analytic near 0

6° Riemann surface:

Σ: topological space with some property



$\forall\, p \in \Sigma$ $\exists$ nbd $\mathcal{U}$ of p and $\varphi: \mathcal{U} \to V$: open set in $\mathbb{C}$
homeomorphism

such that $\varphi \circ \tilde{\varphi}^{-1}: \underbrace{\tilde{\varphi}(\mathcal{U} \cap \tilde{\mathcal{U}})}_{\subset \tilde{V}} \longrightarrow \underbrace{\varphi(\mathcal{U} \cap \tilde{\mathcal{U}})}_{\subset V}$

is a conformal equivalence.

<u>Question</u> analytic / meromorphic functions on Σ?
differentials on Σ?

If Σ_1, Σ_2 are both genus g surface, are they "conformal" equivalent?

<u>rmk</u> $f(z) = (g(z))^{\frac{1}{2}}$

Consider $w^2 = (z-a_1)\cdots(z-a_{2n})$

↳ $Z_0^{2n-2} Z_1^2 = (Z_2 - a_1)\cdots(Z_2 - a_{2n}) \subset \mathbb{CP}^2 = \dfrac{\{(Z_0, Z_1, Z_2) \in \mathbb{C}^3 \setminus \{0\}\}}{(Z_0,Z_1,Z_2) \sim \lambda(Z_0,Z_1,Z_2)}$ $\lambda \in \mathbb{C} \setminus \{0\}$